On the Stability Analysis of Discrete Linear Time–Varying Systems

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Abstract

This article presents a new approach for the exponential stability (ES) analysis of discrete linear time–varying systems (TVSs), which is widely used to study control systems in aerospace engineering. With the introduction of summation function for the discrete linear time–varying systems and satisfying some of its characteristics, a necessary and sufficient condition is obtained for the exponential stability of discrete linear TVSs.

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Introduction

In general control system design deals with the problem of making a concrete physical system working according to some given specifications. Stability is an essential archetype in control system design and mathematical control theory. The main interest is the asymptotic behavior of solutions and different types of stabilities in the study of such systems. Results related to stability of different system can be found in Amato et al. (1993); Berger and Ilchmann (2013); Okano et al. (2006); Trentelman et al. (2002). Stability theory have fundamental role in the field of engineering and physics. Difference equations and differential equations are the main tools for illustrating the process of change over time. It is very difficult to find the solution of those systems explicitly, or it is very difficult to manage the solution of such equations. For well approximate solution various numerical methods can be used at fixed intervals. Also the qualitative behaviors of solutions for such system is an interesting issue for many mathematicians like Wu (1984); Zada et al. (2016, 2017); Zada and Ali (2018).

Stability of linear time–varying system is recursive identification and adaptive control of a random important technical issues. To ensure the stability of time–varying systems in course of recursive identification the predicted inverse system is time–varying and class of inverse systems should be cramped. In stochastic sense finite-dimensional systems with fixed dimension can be used for the interpretation of inverse plants to find

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conditions with the help of which the time varying system exited by a signal which is bounded in the sense of stochastic.

Recently, researchers have made significant progress. There are mainly two types of methods used for the stability analysis of linear time–varying systems, which are frozen time method and parallel D-spectrum method Mullhaupt et al. (2007); Forbes and Damaren (2011). Stability of linear time varying system is discussed by some of the mathematicians. Hill discussed the stability of both continuous and discrete time-varying linear systems, which describes that how the stability estimates are obtained in either case in terms of the Lipschitz constant for the governing matrices and the assumed uniform decay rate of the corresponding frozen time linear systems Rugh (1996); Hill and Ilchmann (2011). Bounds on the exponential growth of continuous/discrete time varying systems have been suggested by numerous authors Desoer (1970); Coppel(1978) and references cited therein.

In this article, we use the idea of summation function (SF) approach to explore the ES of discrete linear time varying systems. In the previous work the author showed the ES of continuous linear time varying systems with the idea of integral function Yao et al. (2012). The main appliance of our approach is the idea of SF, which has some nice properties such as homogeneity, sub-additivity, convexity, common-bound and vertex-bound.

Basic Concepts and Remarks

We consider the discrete system given by

$$\begin{cases}\Theta(\upsilon+1)=B_\upsilon\Theta(\upsilon), \quad \upsilon\in Z^+,\\ \Theta(0)=\Theta_0\end{cases} \tag{1}$$

where B_υ are matrices over R^υ. The solution of such system is given as

$$\Theta(\upsilon,\Theta_0)=B_{\upsilon-1}B_{\upsilon-2}\cdots B_0\Theta_0.$$

Solution

Given that

$$\Theta(\upsilon+1)=B_\upsilon\Theta(0), \qquad \Theta(0)=\Theta_0, \qquad \upsilon\in Z^+,$$

for $n=0$

$$\Theta(1)=B_0\Theta_0, \qquad \Theta(0)=\Theta_0,$$

for $n=1$

$$\Theta(2)=B_1B_0\Theta_0, \qquad \Theta(1)=B_0\Theta_0$$

$$\Theta(\upsilon)=B_{(\upsilon-1)}B_{(\upsilon-2)}B_{(\upsilon-3)}\cdots B_{(0)}\Theta_0$$

$$\Rightarrow \Theta(\upsilon,\Theta_0)=B_{(\upsilon-1)}B_{(\upsilon-2)}\cdots B_{(0)}\Theta_0,$$

which is the solution of the given system.

Definition 1 *If $\Theta(\upsilon, \Theta_0)$ represents the solution of the system (1) with initial condition Θ_0. The system is called exponentially stable if for some $p \in (0,1)$ and $M_p \ni$*

$$||\Theta(\upsilon, \Theta_0)|| \leq M_p p^{\upsilon-\varsigma} ||\Theta_0||, \;\; for\ every \;\; n \geqslant m.$$

Definition 2 *The exponential decay rate for the system (1) is defined as*

$$p^* = inf\{p ||\Theta(\upsilon, \Theta_0)|| \leq M_p p^{\upsilon-\varsigma} ||\Theta_0||, \;\; \Theta_0 \in R^{\upsilon}, n \geqslant m\}.$$

Lemma 3 *The solution of the linear TVS (1) has the following properties,*

$$\Theta(\upsilon, \beta\Theta_0) = \beta\Theta(\upsilon, \Theta_0), \;\; \beta \in R$$

$$\Theta(\upsilon, \Theta_0 + \Theta_1) = \Theta(\upsilon, \Theta_0) + \Theta(\upsilon, \Theta_1), \;\; \Theta_0, \Theta_1 \in R^{\upsilon}$$

Proof. By the system (1) let us consider

$$\begin{cases} \Theta(\upsilon + 1) = B_\upsilon \Theta(\upsilon), \quad \upsilon \in Z^+, \\ \Theta(0) = \beta\Theta_0 \end{cases} \tag{2}$$

Given

$$\Theta(\upsilon + 1) = B_\upsilon \Theta(\upsilon), \qquad \Theta(0) = \beta\Theta_0, \qquad \upsilon \in Z^+$$

for $n = 0$

$$\Theta(1) = B_0 \beta\Theta_0, \qquad \Theta(0) = \beta\Theta_0.$$

for $n = 1$

$$\Theta(2) = B_1 B_0 \beta\Theta_0, \qquad \Theta(1) = B_0 \beta\Theta_0$$

$$\Theta(\upsilon) = B_{(\upsilon-1)} B_{(\upsilon-2)} B_{(\upsilon-3)} \cdots B_0 \beta\Theta_0.$$

The solution of this system is

$$\Rightarrow \Theta(\upsilon, \beta\Theta_0) = B_{(\upsilon-1)} B_{(\upsilon-2)} \cdots B_{(0)} \beta\Theta_0$$

$$\Rightarrow \Theta(\upsilon, \beta\Theta_0) = \beta(B_{(\upsilon-1)} B_{(\upsilon-2)} \cdots B_{(0)} \Theta_0)$$

$$\Rightarrow \Theta(\upsilon, \beta\Theta_0) = \beta\Theta(\upsilon, \Theta_0).$$

And

$$\Theta(\upsilon, \Theta_0 + \Theta_1) = \Theta(\upsilon, \Theta_0) + \Theta(\upsilon, \Theta_1).$$

Consider

$$\Theta(\upsilon + 1) = B_\upsilon \Theta(\upsilon), \quad \upsilon \in Z^+, \quad \Theta(0) = \Theta_0 + \Theta_1$$

for $n = 0$

$$\Theta(1) = B_0(\Theta_0 + \Theta_1), \qquad \Theta(0) = \Theta_0 + \Theta_1$$

for $n = 1$

$$\Theta(2) = B_1 B_0(\Theta_0 + \Theta_1), \qquad \Theta(1) = B_0(\Theta_0 + \Theta_1)$$

$$\Theta(\upsilon) = B_{(\upsilon-1)}B_{(\upsilon-2)}B_{(\upsilon-3)}\cdots B_0(\Theta_0+\Theta_1),$$

also the solution of this system is

$$\Theta(\upsilon,\Theta_0+\Theta_1) = B_{(\upsilon-1)}B_{(\upsilon-2)}B_{(\upsilon-3)}\cdots B_0(\Theta_0+\Theta_1)$$

$$\Rightarrow \Theta(\upsilon,\Theta_0+\Theta_1) = B_{(\upsilon-1)}B_{(\upsilon-2)}\cdots B_0\Theta_0 + B_{(\upsilon-1)}B_{(\upsilon-2)}\cdots B_0\Theta_1$$

$$\Rightarrow \Theta(\upsilon,\Theta_0+\Theta_1) = \Theta(\upsilon,\Theta_0)+\Theta(\upsilon,\Theta_1).$$

Definition 4 (SF) *Consider the function $\mathcal{I}(.,\Theta_0)$ of the linear TVS as,*

$$\mathcal{I}(\tau,\Theta_0) = \sup\sum_{\varsigma=0}^{\infty}\tau^{\upsilon-\varsigma}||\Theta(\upsilon,\Theta_0)||^2.$$

For every fixed point $\tau \geqslant 0$, $\mathcal{I}(\tau,\Theta_0)$ is a relation of Θ_0 only,

$$I_\tau(\Theta_0) = \mathcal{I}(\tau,\Theta_0).$$

Theorem 5 *(Properties of $\mathcal{I}(\tau,\Theta_0)$)*

2.5.1 *Homogeneity*:

$$\mathcal{I}(\tau,\beta\Theta_0) = \beta^2\mathcal{I}(\tau,\Theta_0).$$

2.5.2 Sub-additivity: For all $\Theta_1,\Theta_2 \in R^\upsilon$,

$$\sqrt{\mathcal{I}(\tau,\Theta_1+\Theta_2)} \le \sqrt{\mathcal{I}(\tau,\Theta_1)}+\sqrt{\mathcal{I}(\tau,\Theta_2)}.$$

2.5.3 Convexity: For every $\tau \geqslant 0, \sqrt{\mathcal{I}(\tau,\Theta_1)}$ is convex function, i.e for all $\Theta_1,\Theta_2 \in R^\upsilon$ and $\beta_1,\beta_2 \geqslant 0, \beta_1+\beta_2 = 1$,

$$\sqrt{\mathcal{I}(\tau,\beta_1\Theta_1+\beta_2\Theta_2)} \le \beta_1\sqrt{\mathcal{I}(\tau,\Theta_1)}+\beta_2\sqrt{\mathcal{I}(\tau,\Theta_2)}.$$

2.5.4 *Common-Bound*: For every $\tau \geqslant 0, \mathcal{I}(\tau,\Theta_0) < \infty$ for all $\Theta_0 \in R^\upsilon$

$$\mathcal{I}(\tau,\Theta_0) < c||\Theta_0||^2,$$

for some c.

2.5.5 *Vertex-Bound*:

$$\mathcal{I}(\tau,\Theta_i) < \infty,\ i \in \{1,2,3,\cdots n\},$$

$\Rightarrow\ \mathcal{I}(\tau,\Theta_0) < \infty\ \ \forall\ \ \Theta_0 \in R^\upsilon$, where $\{\Theta_i\}_{i=1}^{\upsilon}$ is standard basis (SB) of R^υ.

Proof. **2.5.1 Homogeneity:** To prove the Homogeneity we use the definition (4),

$$\mathcal{I}(\tau,\Theta_0) = \sup\sum_{\varsigma=0}^{\infty}\tau^{\upsilon-\varsigma}||\Theta(\upsilon,\Theta_0)||^2.$$

Now consider

$$\begin{aligned}\mathcal{I}(\tau,\beta\Theta_0) &= \sup\sum_{\varsigma=0}^{\infty}\tau^{\upsilon-\varsigma}||\Theta(\upsilon,\beta\Theta_0)||^2\\ &= \sup\sum_{\varsigma=0}^{\infty}\tau^{\upsilon-\varsigma}||\beta\Theta(\upsilon,\Theta_0)||^2\\ &= \sup\sum_{\varsigma=0}^{\infty}\tau^{\upsilon-\varsigma}\beta^2||\Theta(\upsilon,\Theta_0)||^2\\ &= \beta^2\sup\sum_{\varsigma=0}^{\infty}\tau^{\upsilon-\varsigma}||\Theta(\upsilon,\Theta_0)||^2\end{aligned}$$

$= \beta^2 \mathcal{J}(\tau, \Theta_0).$

2.5.2 Sub-Additvity: For any $\Theta_0, \Theta_1 \in R^\upsilon$ we need to show that

$$\sqrt{\mathcal{J}(\tau, \Theta_1 + \Theta_2)} \leq \sqrt{\mathcal{J}(\tau, \Theta_1)} + \sqrt{\mathcal{J}(\tau, \Theta_2)}.$$

Now we consider

$$I_\tau(\Theta_1 + \Theta_2) = \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_1 + \Theta_2)||^2$$
$$= \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} \{||\Theta(\upsilon, \Theta_1 + \Theta_2)||^2\}$$
$$= \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} \{||\Theta(\upsilon, \Theta_1)||^2 + ||\Theta(\upsilon, \Theta_2)||^2| + 2||\Theta(\upsilon, \Theta_1)||||\Theta(\upsilon, \Theta_2)||\}$$
$$= \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_1)||^2 + \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_2)||^2$$
$$+ \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} 2||\Theta(\upsilon, \Theta_1)||||\Theta(\upsilon, \Theta_2)||$$
$$\leq \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_1)||^2 + \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_2)||^2$$
$$+ 2\sqrt{\sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_1)||} \sqrt{\sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_2)||}$$
$$\leq \mathcal{J}(\tau, \Theta_1) + \mathcal{J}(\tau, \Theta_2) + 2\sqrt{\mathcal{J}(\tau, \Theta_1)}\sqrt{\mathcal{J}(\tau, \Theta_2)}$$
$$\leq (\sqrt{\mathcal{J}(\tau, \Theta_1)})^2 + (\sqrt{\mathcal{J}(\tau, \Theta_2)})^2 + 2\sqrt{\mathcal{J}(\tau, \Theta_1)}\sqrt{\mathcal{J}(\tau, \Theta_2)}$$
$$= \{\sqrt{\mathcal{J}()\tau, \Theta_1} + \sqrt{\mathcal{J}(\tau, \Theta_2)}\}^2,$$

taking square root

$$\sqrt{\mathcal{J}(\tau, \Theta_1 + \Theta_2)} \leq \sqrt{\mathcal{J}(\tau, \Theta_1)} + \sqrt{\mathcal{J}(\tau, \Theta_2)}.$$

2.5.3 Convexity:

For every $\tau \geqslant 0, \sqrt{\mathcal{J}(\tau, \Theta_1)}$ is convex function for all $\Theta_1, \Theta_2 \in R^\upsilon$ and $\beta_1, \beta_2 \geqslant 0, \beta_1 + \beta_2 = 1,$ we have

$$\sqrt{\mathcal{J}(\tau, \beta_1\Theta_1 + \beta_2\Theta_2)} \leq \beta_1\sqrt{\mathcal{J}(\tau, \Theta_1)} + \beta_2\sqrt{\mathcal{J}(\tau, \Theta_2)}.$$

By definition (4)

$$\mathcal{J}(\tau, \beta_1\Theta_1 + \beta_2\Theta_2) = \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \beta_1\Theta_1 + \beta_2\Theta_2)||^2$$

$$= \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} \{||\Theta(\upsilon, \beta_1\Theta_1) + \Theta(\upsilon, \beta_2\Theta_2)||^2\},$$

$using\ \ lemma(3$

$$= \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} \{||\Theta(\upsilon, \beta_1\Theta_1)||^2 + ||\Theta(\upsilon, \beta_2\Theta_2)||^2$$

$$+ 2||\Theta(\upsilon, \beta_1\Theta_1)||||\Theta(\upsilon, \beta_2\Theta_2)||\}$$

$$= \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \beta_1\Theta_1)||^2 +$$

$\sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma}||\Theta(\upsilon,\beta_2\Theta_2)||^2$

$$+\sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma}2||\Theta(\upsilon,\beta_1\Theta_1)||||\Theta(\upsilon,\beta_2\Theta_2)||$$

$$\leq \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma}\beta_1^2||\Theta(\upsilon,\Theta_1)||^2 + \sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma}\beta_2^2||\Theta(\upsilon,\Theta_2)||^2$$

$$+2\sqrt{\sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma}\beta_1^2||\Theta(\upsilon,\Theta_1)||} + \sqrt{\sup \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma}\beta_2^2||\Theta(\upsilon,\Theta_2)||}$$

$$\leq (\sqrt{\beta_1^2\mathcal{I}(\tau,\Theta_1)})^2 + (\sqrt{\beta_2^2\mathcal{I}(\tau,\Theta_2)})^2 + 2\sqrt{\beta_1^2\mathcal{I}(\tau,\Theta_1)}\sqrt{\beta_2^2\mathcal{I}(\tau,\Theta_2)}$$

$$\leq \{\sqrt{\beta_1^2\mathcal{I}(\tau,\Theta_1)} + \sqrt{\beta_2^2\mathcal{I}(\tau,\Theta_2)}\}^2$$

$$\mathcal{I}(\tau,\beta_1\Theta_1+\beta_2\Theta_2) \leq \{\beta_1\sqrt{\mathcal{I}(\tau,\Theta_1)} + \beta_2\sqrt{\mathcal{I}(\tau,\Theta_2)}\}^2,$$

now taking square root both side

$$\sqrt{\mathcal{I}(\tau,\beta_1\Theta_1+\beta_2\Theta_2)} \leq \beta_1\sqrt{\mathcal{I}(\tau,\Theta_1)} + \beta_2\sqrt{\mathcal{I}(\tau,\Theta_2)}.$$

2.5.4 Common Bound: Consider $\tau \geqslant 0 \ni I_\tau(y) < \infty$. Suppose $\{\Theta_i\}_{i=1}^{\upsilon}$ denote a SB of R^υ, then for every $y \in S^{\upsilon-1}$, we can find $\beta_i \geqslant 0$, for $\sum_{i=1}^{\upsilon} \beta_i^2 = 1$, $\ni$

$$y = \sum_{i=1}^{\upsilon} \beta_i\Theta_i.$$

Using the property of sub-additivity of $I_\tau(y)$ and Cauchy-Schwartz (CS) inequality in the summation form, for any $y \in S^{\upsilon-1}$ we get that,

$$I_\tau(y) = \sqrt{I_\tau(\sum_{i=1}^{\upsilon} \beta_i\Theta_i)}$$

using the homogeneity property

$$\leq [\sum_{i=1}^{\upsilon} \beta_i\sqrt{I_\tau\Theta_i}]^2,$$

$$\leq \sum_{i=1}^{\upsilon} \beta_i^2 \sum_{i=1}^{2} I\tau(\Theta_i)$$

$$\leq c.$$

Thus we have

$$I_\tau(\Theta) < c||\Theta||^2, y \in R^\upsilon.$$

2.5.5 Vertex Bound: We have

$\mathcal{I}(\tau,\Theta_i) < \infty, i \in \{1,2,\ldots\ldots.n\}, \Rightarrow \mathcal{I}(\tau,\Theta_0) < \infty,$ for all $\Theta_0 \in R^\upsilon$,

where $\{\Theta_i\}_{i=1}^{\upsilon}$ is a standard bases of R^{υ}.
Consider

$$y = \sum_{i=1}^{\upsilon} \beta_i \Theta_i.$$

Using the property of sub-additivity of $I_\tau(y)$ and CS inequality in the summation form, for any $y \in S^{\upsilon-1}$ we get that,

$$I_\tau(y) = \sqrt{I_\tau(\sum_{i=1}^{\upsilon} \beta_i \Theta_i)}$$

$$\leq [\sum_{i=1}^{\upsilon} \beta_i \sqrt{I_\tau \Theta_i}]^2$$

$$\leq \sum_{i=1}^{\upsilon} \beta_i^2 \sum_{i=1}^{2} I\tau(\Theta_i)$$

$$\leq c < \infty.$$

Exponential Stability of Discrete Linear Time Varying Systems

In this section we will discuss a sufficient and necessary condition\of the ES of linear TVS (1) with the help of defined function (4).

Theorem 6 *Consider the linear TVS (1) the following statements are equivalent.*

1. The TVS (1) is exponentially stable.
2. There exists $\tau > 1$ and a SB $\{\Theta_i\}_{i=1}^{\upsilon}$ $\ni$ the defined function is bounded i.e,

$$I_\tau(\Theta_i) < \infty, \ i \in \{1,2,\cdots,n\}.$$

3. $\exists\ \tau > 1 \ni I_\tau$ is bounded for every $y \in R^{\upsilon}$.

Proof. $(\mathbf{1}) \longrightarrow (\mathbf{2})$: Suppose that the linear TVS (1) is exponential stable, i.e, we can find constants $p \in (0,1)$ and $M_p \ni$

$$||\Theta(\upsilon, \Theta_0)|| \leq M_p p^{\upsilon-\varsigma} ||\Theta_o||,$$

for every $n \geqslant m.$ By using the condition we can write that

$$I_\tau(\Theta_0) = \sup_{\varsigma \geqslant 0} \sum_{\varsigma}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\upsilon, \Theta_0)||^2$$

$$\leq \sup_{\varsigma \geqslant 0} \sum_{\varsigma}^{\infty} M_p^2 (\tau p^2)^{\upsilon-\varsigma} ||\Theta_0||^2$$

$$= \sum_{0}^{\infty} M_p^2 (\tau p^2)^{\upsilon} ||\Theta_o||^2$$

$$< \infty.$$

$(\mathbf{2}) \longrightarrow (\mathbf{3})$: Consider $\tau \geqslant 1 \ni I_\tau(y) < \infty$. Suppose $\{\Theta_i\}_{i=1}^{\upsilon}$ be a SB of

R^{υ}, then for every $y \in S^{\upsilon-1}$, we can find $\beta_i \geqslant 0$, for $\sum_{i=1}^{\upsilon} \beta_i^2 = 1$, $\ni$

$$y = \sum_{i=1}^{\upsilon} \beta_i \Theta_i.$$

Using the property of sub-additivity of $I_\tau(y)$ and CS inequality in the summation form, for any $y \in S^{\upsilon-1}$ we get that,

$$I_\tau(y) = \sqrt{I_\tau(\sum_{i=1}^{\upsilon} \beta_i \Theta_i)}$$

$$\leq [\sum_{i=1}^{\upsilon} \beta_i \sqrt{I_\tau \Theta_i}]^2$$

$$\leq \sum_{i=1}^{\upsilon} \beta_i^2 \sum_{i=1}^{2} I\tau(\Theta_i)$$
$$\leq c.$$

Thus by using homogeneity property we have $I_\tau(y) < c||y||^2, y \in R^{\upsilon}$.

$(\mathbf{3}) \longrightarrow (\mathbf{1})$: Let $\tau \geqslant 1 \ \ni \ I_\tau(\Theta_0)$ is bounded. By the statement 4 in theorem (5) we learn that, $\exists \ c > 0 \ \ni$

$$\mathcal{J}(\tau, \Theta_0) = sup_{\upsilon \geqslant 0} \sum_{\varsigma=0}^{\infty} \tau^{\upsilon-\varsigma} ||\Theta(\varsigma, \Theta_0)||^2 \leq c. \tag{3}$$

This implies that , For any given $\in > 0$, then $\exists \ l_0 > 0 \ \ni$ for all $m \geqslant 0$,

$$\sum_{\varsigma=l}^{l+k} \tau^{\upsilon-\varsigma} ||\Theta(\varsigma, \Theta_0)||^2 < \epsilon, \quad l > l_0, \tag{4}$$

using the summation of (4) we have that , $\exists \ n^* < n \ \ni$

$$\sum_{\varsigma=1}^{l+k} \tau^{\upsilon-\varsigma} ||\Theta(\varsigma, \Theta_0)||^2 = \tau^{\upsilon-l} ||\Theta(l, \Theta_0)||^2 + \tau^{\upsilon-l-1} ||\Theta(l + 1, \Theta_0)||^2 + \cdots$$
$$+ \tau^{\upsilon-l-k} ||\Theta(l + k, \Theta_0)||^2$$

$$= \tau^{\upsilon-l} [||\Theta(l, \Theta_0)||^2 + \frac{1}{\tau} ||\Theta(l + 1, \Theta_0)||^2 + \cdots + \frac{1}{\tau^k} ||\Theta(l + k, \Theta_0)||^2],$$

let $n - l = n^* \Rightarrow l = n - n^*$, thus

$$\sum_{\varsigma=l}^{l+k} \tau^{\upsilon-\varsigma} ||\Theta(\varsigma, \Theta_0)||^2 = \tau^{\upsilon^*} [1 + \frac{1}{\tau} + \cdots + \frac{1}{\tau^k}] \text{sup} ||\Theta(\upsilon^*, \Theta_0)||^2,$$

where $n^* > l + k$.

Taking $\lim_{k \longrightarrow \infty,}$ we get

$$\sum_{\varsigma=l}^{l+k} \tau^{\upsilon-\varsigma} ||\Theta(\varsigma, \Theta_0)||^2 \leq \tau^{\upsilon^*} \text{sup} ||\Theta(\upsilon^*, \Theta_0)||^2 < \epsilon. \tag{5}$$

On the other hand, for all $n \geqslant 0$,

$$||\Theta(\upsilon, \Theta_0)||^2 = \sum_{\varsigma=n^*}^{\upsilon} \Delta(||\Theta(\varsigma, \Theta_0)||^2) + ||\Theta(\upsilon^*, \Theta_0)||^2$$

$$= \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma+1,\Theta_0)||^2 - \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2$$

$$= \sum_{\varsigma=n^*}^{\upsilon} ||\beta_\varsigma\Theta((\varsigma),\Theta_0)||^2 - \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2$$

$$\leq \sum_{\varsigma=n^*}^{\upsilon} ||\beta_\varsigma||^2||\Theta(\varsigma+1,\Theta_0)||^2 - \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2$$

$$\leq (c^2-1)\sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2,$$

where c is the upper bound of $||\beta_{(.)}||$

$$\leq (c^2 - 1(\upsilon - n^*)\sup||\Theta(\upsilon^*,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2$$

$$\leq (l(c^2-1)\sup||\Theta(\upsilon^*,\Theta_0)||^2.$$

From equation (5) we get

$$||\Theta(\upsilon,\Theta_0)||^2 \leq ((c^2-1)l+1)\tau^{-n^*}\epsilon$$

$$\leq ((c^2-1)l+1)\tau^{l-n^*}\epsilon$$

let $M_p = ((c^2-1)l+1)\epsilon$, then

$$||\Theta(\upsilon,\Theta_0)||^2 \leq M_p\tau^{-(\upsilon-l)}.$$

At last we obtained the result that the discrete system (1) is exponentially stable.

Definition 7 *The radius of the convergence of* $\mathcal{I}(\tau,\Theta_0)$ *denoted by* τ^*, *is:*

$$\tau^* = \sup\{\tau \geqslant \frac{1}{\mathcal{I}(\tau,\Theta_0)} < \infty,\ \forall\ \Theta_0 \epsilon Z^{\upsilon}\}.$$

Theorem 8 *Given the linear TVS (1) with* τ^* *then for any* $p > (\tau^*)^{-\frac{1}{2}}$, $\exists$ *a constant* M_p $\ni$

$$||\Theta(\upsilon,\Theta_0)|| \leq M_p p^{\upsilon-\varsigma}||\Theta_0||, n \geqslant m \geqslant 0.$$

Furthermore

$$\tau^* = (\tau^*)^{-\frac{1}{2}}.$$

Proof. Suppose that the linear TVS (1) is exponential stable, i.e, we can find constants M_p and $p \geqslant (\tau^*)^{-\frac{1}{2}}$ $\ni$

$$||\Theta(\upsilon,\Theta_0)|| \leq M_p p^{\upsilon-\varsigma}||\Theta_o||,$$

for every $n \geqslant m$. By using the condition we can write that

$$I_\tau(\Theta_0) = \sup_{\varsigma\geqslant 0}\sum_{\varsigma}^{\infty} \tau^{\upsilon-\varsigma}||\Theta(\upsilon,\Theta_0)||^2.$$

Here $\tau^* = (\tau^*)^{-\frac{1}{2}}$

$$I_\tau(\Theta_0) = \sup_{\varsigma\geqslant 0}\sum_{\varsigma}^{\infty} \tau^{-\frac{1}{2}(\upsilon-\varsigma)}||\Theta(\upsilon,\Theta_0)||^2$$

$$I_\tau(\Theta_0) = \sup_{\varsigma\geqslant 0}\sum_{\varsigma}^{\infty} M_p^2(\tau^{-\frac{1}{2}}p^2)^{\upsilon-\varsigma}||\Theta_0||^2$$

$$= \sum_0^\infty M_p^2(\tau^{-\frac{1}{2}}p^2)^\upsilon ||\Theta_o||^2 < \infty.$$

Now assume $\tau^* = (\tau^*)^{-\frac{1}{2}} \ni I_\tau(\Theta_0)$ is bounded. Now $\exists\ c > 0 \ni$

$$\mathcal{J}(\tau, \Theta_0) = sup_{\upsilon \geqslant 0} \sum_{\varsigma=0}^\infty \tau^{-\frac{1}{2}(\upsilon-\varsigma)}||\Theta(\varsigma, \Theta_0)||^2 \leq c. \quad (6)$$

This implies that , For any given $\epsilon > 0,$ then $\exists\ l_0 > 0 \ni$ for all $m \geqslant 0,$

$$\sum_{\varsigma=l}^{l+k} \tau^{-\frac{1}{2}(\upsilon-\varsigma)}||\Theta(\varsigma, \Theta_0)||^2 < \epsilon, \ \ l > l_0, \quad (7)$$

using the summation of (7) we have that , $\exists\ n^* < n \ni$

$$\sum_{\varsigma=l}^{l+k} \tau^{-\frac{1}{2}(\upsilon-\varsigma)}||\Theta(\varsigma, \Theta_0)||^2 = \tau^{-\frac{1}{2}(\upsilon-l)}||\Theta(l, \Theta_0)||^2 + \tau^{-\frac{1}{2}(\upsilon-l-1)}||\Theta(l+1, \Theta_0)||^2 + \cdots + \tau^{-\frac{1}{2}(\upsilon-l-k)}||\Theta(l+k, \Theta_0)||^2$$

$$= \tau^{-\frac{1}{2}(\upsilon-l)}[||\Theta(l, \Theta_0)||^2 + \tau^{\frac{1}{2}}||\Theta(l+1, \Theta_0)||^2 + \cdots + \tau^{\frac{k}{2}}||\Theta(l+k, \Theta_0)||^2],$$

let $-\frac{1}{2}(\upsilon - l) = n^* \Rightarrow l - n = 2n^*,$ thus

$$\sum_{\varsigma=l}^{l+k} \tau^{-\frac{1}{2}(\upsilon-\varsigma)}||\Theta(\varsigma, \Theta_0)||^2 = \tau^{\upsilon^*}[1 + \tau^{\frac{1}{2}} + \tau + \cdots + \tau^{\frac{k}{2}}]\sup||\Theta(\upsilon^*, \Theta_0)||^2, \ where\ n^* > l + k.$$

Taking $\lim_{k\to\infty,}$ we get

$$\sum_{\varsigma=l}^{l+k} \tau^{-\frac{1}{2}(\upsilon-\varsigma)}||\Theta(\varsigma, \Theta_0)||^2 \leq \tau^{\upsilon^*}\sup||\Theta(\upsilon^*, \Theta_0)||^2 < \epsilon. \quad (8)$$

On the other hand, for all $n \geqslant 0,$

$$||\Theta(\upsilon, \Theta_0)||^2 = \sum_{\varsigma=n^*}^{\upsilon} \Delta(||\Theta(\varsigma, \Theta_0)||^2) + ||\Theta(\upsilon^*, \Theta_0)||^2,$$

apply Δ operator

$$= \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma+1, \Theta_0)||^2 - \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma, \Theta_0)||^2 + ||\Theta(\upsilon^*, \Theta_0)||^2,$$

by the equation (1) take system matrix $\frac{\beta_(\upsilon)}{r}$

$$= \sum_{\varsigma=n^*}^{\upsilon} ||\frac{\beta_(\upsilon)}{r}\Theta((\varsigma), \Theta_0)||^2 - \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma, \Theta_0)||^2 + ||\Theta(\upsilon^*, \Theta_0)||^2$$

$$\leq \sum_{\varsigma=n^*}^{\upsilon} ||\frac{\beta_(\upsilon)}{r}||^2||\Theta(\varsigma+1, \Theta_0)||^2 - \sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma, \Theta_0)||^2 + ||\Theta(\upsilon^*, \Theta_0)||^2$$

$$\leq (c^2-1)\sum_{\varsigma=n^*}^{\upsilon} ||\Theta(\varsigma,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2,$$

where c is the upper bound of $||\frac{\beta_{(\upsilon)}}{r}||$

$$\leq (c^2-1(2n^*-n)\sup||\Theta(\upsilon^*,\Theta_0)||^2 + ||\Theta(\upsilon^*,\Theta_0)||^2$$

$$\leq (l(c^2-1)+1)\sup||\Theta(\upsilon^*,\Theta_0)||^2.$$

From equation (8) we get

$$||\Theta(\upsilon,\Theta_0)||^2 \leq ((c^2-1)l+1)\tau^{-n^*}\epsilon$$
$$\leq ((c^2-1)l+1)\tau^{\frac{1}{2}(\upsilon-l)}\epsilon,$$

let $M_p=((c^2-1)l+1)\epsilon$, then

$$||\Theta(\upsilon,\Theta_0)||^2 \leq M_p\tau^{\frac{1}{2}(\upsilon-l)}.$$

So the system is exponentially stable.

Definition 9 *For each* $j>0$, *define*

$$I^j(\tau,\Theta_0) = \sup\sum_{\varsigma=0}^{j} \tau^{\upsilon-\varsigma}||\Theta(\varsigma,\Theta_0)||^2.$$

By the same process, we can prove that

Proposition 10 *Consider the Linear TVS (1), then*

1. For $\tau>0$, $\mathcal{I}(.,\Theta_0)$ is unbounded $\forall\ \Theta_0\in Z^{\varsigma}$, then $I^j(\tau,\Theta_0)$ will also be unbounded by the increasing of j.

2. For $\tau>0$, $\mathcal{I}(.,\Theta_0)$ is bounded $\forall\ \Theta_0\in Z^{\varsigma}$, then

$$lim_{j\rightarrow\infty}I^j(\tau,\Theta_0)=\mathcal{I}(\tau,\Theta_0).$$

Proof. 1. Assume that $\exists\ \Theta_0\epsilon Z^{\upsilon}, \mathcal{I}(\tau,\Theta_0)$ is bounded, but $I^j(\tau,\Theta_0)$ is bounded in $j\epsilon Z^+$, i.e, $\exists\ c>0\ \ni$ for all $\Theta_0\epsilon S^{\upsilon-1}, I^j(\tau,\Theta_0)\leq c$, for all $j\geqslant 0$. This implies that, for all $n\epsilon[0,j]$ and $\Theta_0\epsilon S^{\upsilon-1}$,

$$\sum_{\varsigma=j}^{j+k} \tau^{\upsilon-\varsigma}||\Theta(\varsigma,\Theta_0)||^2 \leq c,\ \ j>j_0.$$

By the similar deduction with the proof for $(3)\longrightarrow(1)$ in theorem (1) we learn that , $\exists$ constant $M_p>0\ \ni$

$$||\Theta(\varsigma,\Theta_0)|| < M_p\tau^{-\frac{1}{2}(\upsilon-l)}, n>l.$$

From lemma (3) the above condition implies that $\tau^{-\frac{1}{2}} > (\tau^*)^{-\frac{1}{2}}$, which further yields $\tau<\tau^*$. However, this is in contradiction with the assumption that $I_\tau(\Theta_0)$ is unbounded.

2. For $\tau<\tau^*$, let $\tau_A=(\tau<\tau^*)/2$, then we have

$$I\tau_A(\Theta_0)<\infty, Y_0\epsilon Z^{\upsilon}.$$

It can be learn from theorem 2, there exist constant $M>0\ \ni$

$$||\Theta(\varsigma,\Theta_0)|| < M\tau_A^{\frac{-1}{2(\upsilon-l)}}, n>j.$$

With the help of basic inequality we have

$$||I_\tau^j - I\tau_1(\Theta_0)|| \le || \sum_{\varsigma=j}^{\infty} \tau^\varsigma ||\Theta(\varsigma, \Theta_0)||^2||$$

$$\le M^2 \sum_{\varsigma=j}^{\infty} \tau^\varsigma \tau_A^{-(\varsigma-l)}$$

$$\le M^2 \sum_{\varsigma=j}^{\infty} (\frac{\tau}{\tau_A^{-l}})^\varsigma.$$

Note that $0 < \tau < \tau_A$, then letting j converge to infinity we have for all $Y_0 \epsilon Z^\upsilon$,

$$lim_{l\to\infty} I_\tau^j(\Theta_0) = I_\tau(\Theta_0).$$

Conclusion

In this paper, we studied the ES of discrete linear TVS using SF.By showing the properties of SF and applying the lemma to derive the basic properties for the ES of discrete linear TVS. Furthermore, the exponential decay rate of the system can be obtained by computing the radii of convergence of SF. In future, our aim is to extend these results to the Hyers–Ulam stability of discrete linear TVS using SF.

Competing Interests

The authors declare that they have no competing interests.

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