On Penta Topological Separation Axioms

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Abstract

In the past few decades separation axioms in single topological spaces have been extended to bi, tri and quad topological spaces by many authors. In this work we examine separation axioms in penta topological spaces and also investigate some of their properties.

***Keywords**:* penta topological separation axioms, $\wp T_0$, $\wp T_1$, $\wp T_2, \wp T_3, \wp regular$.

Introduction

In general topology separation axioms are topological spaces telling us how distinct points, points and sets and sets are wrapped by open sets (or neighbourhoods). Among these the most common separation axioms are T_0, T_1, T_2, T_3 and regular axioms. Khan & Khan (2018) introduced the concept of penta topological space as a non-void set equipped with five topologies. To be more specific, if X is a non-void set and $\tau_p = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ is a 5-tuple of topologies, then (X, τ_p) is a penta topological space ($\wp TS$). The 5-tuple of topologies $\tau_p = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ is called penta topology ($\wp T$) on X.

If $\tau_p = (\tau_1, \tau_2, \tau_3, \tau_4)$, then (X, τ_p) is called quad topological space (Mukundan, 2013). Tapi et al (2014) studied separation axioms in quad topological spaces.

If $\tau_p = (\tau_1, \tau_2, \tau_3)$, then (X, τ_p) is called tri topological space. Kovar (2000) introduced the notion of tri topological space. Hameed & Abid (2011) discussed separation axioms in tri-topological spaces.

If $\tau_p = (\tau_1, \tau_2)$, then (X, τ_p) is termed as bi topological space (Kelly, 1963). Kelly was the first who initiated the concept of bi topological space. His work gave birth to a new area of research *viz.* a set with many topologies within the domain of general topology. In bi topological spaces separation axioms in different contexts have been studied by interested authors like (Reilly 1972, Selvanayaki & Rajesh, 2011).

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If $\tau_p = \tau$, where τ is any topology on X, then (X, τ_p) is the classical topological space. For discussion pertaining to separation axioms in classical topological spaces we, however, refer to some standard books available on the topic (Munkres 2000, Simmons 1963 & Willard 1970).

In this paper, we aim at to carry over the idea of separation axioms to $\wp TS$ via penta open sets and penta closed sets (Khan & Khan, 2018). More recently, some new topological concepts in $\wp TS$ have also been discussed by some authors (Anjaline & Pricilla, 2020, Pacifica & Fatima 2019).

Preliminaries

Here we mention some facts about penta topological spaces ($\wp TSs$) which are needed in the sequel. For brevity sake we use X to denote a $\wp TS(X, \tau_p)$.

Definition 2.1

Let X be a $\wp TS$ and S a subset of X. Then

(a) elements of τ_i; $i = 1, 2, 3, 4, 5$ are said to be τ_i-open sets and their relative complements are called τ_i-closed sets

(b) S is called penta open set ($\wp OS$) if $S \epsilon \cup_{i=1}^{5} \tau_i$ and its complement $X \backslash S$ is known as penta closed set ($\wp CS$)

(c) the penta closure of S denoted by $\wp Cl(S)$ is defined as $\wp Cl(S)$ = $\cap_{\lambda \epsilon I} \{C_\lambda$: each C_λ is a $\wp CS$ in X containing $S\}$

(d) S is referred to as penta neighbourhood ($\wp N$) of an element $a \in X$ if and only if there exists a $\wp OS$ Usatisfying the condition $a \in U \subseteq S$.

Theorem 2.2

In $\wp TS$ X

1. $\cup_{\lambda \epsilon I} U_\lambda$ is $\wp OS$ where each U_λ is $\wp OS$
2. $\cap_{\lambda \epsilon I} C_\lambda$ is $\wp CS$ where each C_λ is $\wp CS$

Remark 2.3

Clearly, $\wp Cl(S)$ is a $\wp CS$ set and $S \subseteq \wp Cl(S)$

Theorem 2.4 If X is a $\wp TS$ and $S \subseteq X$, then S is $\wp CS$ if and only if S = $\wp Cl(S)$.

Penta Topological Separation Axioms

We shall use the concepts of $\wp OSs$ and $\wp CSs$ in $\wp TSs$ to introduce penta topological separation axioms ($\wp TSAs$). Some results concerning $\wp TSAs$ are also investigated.

Definition 3.1

A $\wp TS\ X$ satisfies

(1) *Axiom* $\wp T_0$ if and only if for all a and b in $X, a \neq b$, there is a $\wp OS$ U which contains one of a and b and not the other.

(2) *Axiom* $\wp T_1$ if and only if for all a and b in $X, a \neq b$, there are $\wp OSs$ U and V in X such that $a \in U, b \notin U$ and $b \in V, a \notin V$.

(3) *Axiom* $\wp T_2$ if and only if for all a and b in $X, a \neq b$, there are $\wp OSs$ U and V in X such that $a \in U, b \in V$ and $U \cap V = \phi$.

(4) *Axiom of* $\wp regularity$ if and only if for all $\wp CSs$ S in X and $a \notin S$ there are $\wp OSs$ U and V in X such that $S \subseteq U, a \in V$ and $U \cap V = \phi$.

(5) *Axiom* $\wp T_3$ if and only if X obey the axiom of $\wp T_1$ and $\wp$regularity.

A $\wp TS\ X$ which satisfies axiom $\wp T_i; i = 0, 1, 2, 3$ is known as $\wp T_i$ *space*. A $\wp TS$ which satisfies the axiom of $\wp$regularity is called a $\wp$*regular space*. A $\wp T_2$ space is also termed as $\wp Hausdorff$ space.

Examples

Here we give some examples to illustrate $\wp TSs$.

(a) Let $X = \{\alpha, \beta, \gamma, \delta\}$ with $\wp Ts$ $\tau_1 = \{\phi, \{\alpha\}, X\}, \tau_2 = \{\phi, \{\beta\}, \{\alpha, \beta\}, X\}$, $\tau_3 = \{\phi, \{\gamma\}, X\}, \tau_4 = \{\phi, \{\delta\}, X\}$ and $\tau_5 = \{\phi, \{\gamma, \delta\}, X\}$ be a $\wp TS$. The $\wp OSs$ are ϕ, $\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{\alpha, \beta\}, \{\gamma, \delta\}$ and X. Hence X is $\wp T_0$ space.

(b) Let $X = \{\alpha, \beta, \gamma, \delta\}$ with $\wp Ts$ $\tau_1 = \{\phi, \{\alpha\}, X\}, \tau_2 = \{\phi, \{\alpha\}, \{\alpha, \beta\}, X\}$, $\tau_3 = \{\phi, \{\beta\}, \{\beta, \gamma\}, X\}$, $\tau_4 = \{\phi, \{\gamma\}, X\}$ and $\tau_5 = \{\phi, \{\delta\}, X\}$ be a $\wp TS$. The $\wp OSs$ are ϕ, $\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{\alpha, \beta\}, \{\beta, \gamma\}$ and X. Hence X is $\wp T_1$ space.

(c) Let $X = \{\alpha, \beta, \gamma, \delta\}$ with $\wp Ts$ $\tau_1 = \{\phi, \{\alpha\}, X\}, \tau_2 = \{\phi, \{\beta\}, X\}, \tau_3 = \{\phi, \{\gamma\}, X\}$
$\tau_4 = \{\phi, \{\delta\}, X\}, \tau_5 = \{\phi, \{\alpha, \beta\}, X\}$ be a $\wp TS$. The $\wp OSs$ are ϕ, $\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}$,
$\{\alpha, \beta\}$ and X. Hence X is $\wp T_2$ ($\wp$Hausdorff) space.

(d) Let $X = \{\alpha, \beta, \gamma, \delta\}$ with $\wp Ts$ $\tau_1 = \{\phi, \{\alpha, \beta\}, X\}, \tau_2 = \{\phi, \{\beta, \gamma\}, X\}$, $\tau_3 = \{\phi, \{\gamma, \delta\}, X\}, \tau_4 = \{\phi, \{\alpha, \beta\}, \{\gamma, \delta\}, X\}$ and $\tau_5 = \{\phi, X\}$ be a $\wp TS$. The $\wp OSs$ are $\phi, \{\alpha, \beta\}, \{\beta, \gamma\}, \{\gamma, \delta\}, X$ and $\wp CSs$ are $X, \{\gamma, \delta\}, \{\alpha, \delta\}, \{\alpha, \beta\}, \phi$. Hence X is $\wp$regular space.

(e) Consider X with $\wp Ts$ as indicated in (d) above. X is $\wp$regular space and also $\wp T_1$ space. Hence X is $\wp T_3$ space.

Remark

The following implication holds true for $\wp TSAs$:

Axiom $\wp T_3 \Longrightarrow$ Axiom $\wp T_2 \Longrightarrow$ Axiom $\wp T_1 \Longrightarrow$ Axiom $\wp T_0$.

However, the converse of this implication is not true in general.

Main Results

In this section we prove some results concerning $\wp TSAs$ in $\wp TSs$. We start with stating the following fundamental theorem, proof of which is easy and thus omitted.

Theorem 5.1

Let X be a $\wp TS$ and S a subset of X. Then S is $\wp OS$ if and only if it is a $\wp N$of each of its point.

Theorem 5.2

If $\{a\}$ is a $\wp OS$ for some $a \in X$, then $a \notin \wp Cl\{b\}$for every $b \neq a$.

Proof.

Suppose $\{a\}$ is a $\wp OS$ for some a in X, then $X\backslash\{a\}$ is a $\wp CS$ and $a \notin X\backslash\{a\}$.

Assume that $a \in \wp Cl\{b\}$for some $b \neq a$, then b and a both belong to the $\wp CSs$ containing b. It then follows that $a \in X\backslash\{a\}$ contradicting the fact that $a \notin X\backslash\{a\}$.

Hence $a \notin \wp Cl\{b\}$.

Theorem 5.3

In $\wp TS$ X distinct points have distinct $\wp Cls$.

Proof

Consider any two points a, b in X, $a \neq b$ and let $S = X\backslash\{a\}$. Then

$$\wp Cl(S) = S \text{ or } X.$$

If $\wp Cl(S) = S$, then S is $\wp CS$ and so $X\backslash S = \{a\}$ is $\wp OS$ for some a in X such that $b \notin X\backslash S$. By Theorem (5.2) $a \notin \wp Cl\{b\}$ and $b \in \wp Cl\{a\}$ which implies $\wp Cl\{a\}$ and $\wp Cl\{b\}$ are distinct.

If $\wp Cl(S) = X$, then S is $\wp OS$ and so $\{a\}$ is $\wp CS$ which means that $\wp Cl\{a\} = \{a\}$. Thus $\wp Cl\{a\}$ and $\wp Cl\{b\}$ are distinct.

Theorem 5.4

In $\wp TS$ X if distinct points have distinct $\wp Cls$, then it is a $\wp T_0$ space.

Proof. Suppose a and b are two distinct points in X so that $\wp Cl\{a\} \neq \wp Cl\{b\}$. There exists c in X such that

$$c \in \wp Cl\{a\} \text{ but } c \notin \wp Cl\{b\}$$

or

$$c \in \wp Cl\{b\} \text{but } c \notin \wp Cl\{a\}$$

Consider $c \in \wp Cl\{b\}$, then $\wp Cl\{a\}$ is contained in $\wp Cl\{b\}$ which implies $c \notin \wp Cl\{b\}$, a contradiction, so $c \notin \wp Cl\{b\}$. It then follows $c \in \wp Cl$ $(X\backslash\{b\})$. Consequently, X is a $\wp T_0$ space.

Theorem 5.5

Every $\wp TS$ is a $\wp T_0$ space.

Proof.

Immediate from Theorem (5.3) and Theorem (5.4).

Theorem 5.6

A $\wp TS$ X is $\wp T_1$ space if and only if every one-point set in X is $\wp CS$.

Proof. Suppose $\{a\}$ is a one-point set in X and $b \in X\backslash\{a\}$. Then $b \neq a$. Since X is $\wp T_1$ space, so there are $\wp OSs$ U and V in X such that $a \in U, b \notin U$ and $b \in V, a \notin V$. It follows that $b \in V$ and $V \subseteq X\backslash\{a\}$ and so $X\backslash\{a\}$ is a $\wp N$ of b. But b was chosen arbitrary in X, hence $X\backslash\{a\}$ is a $\wp N$ of each of its points. Consequently, $X\backslash\{a\}$ is $\wp OS$ and $\{a\}$ is $\wp CS$.

Conversely, suppose every one point set in X is $\wp CS$. Let $a, b \in X, a \neq b$. Then the one-point sets $\{a\}, \{b\}$ are $\wp CSs$ and so $X\backslash\{a\}, X\backslash\{b\}$ are $\wp OSs$ such that $b \in X\backslash\{a\}, a \notin X\backslash\{a\}$ and
$a \in X\backslash\{b\}, b \notin X\backslash\{b\}$. Hence X is a $\wp T_1$ space.

Corollary 5.7

A $\wp TS$ X is $\wp T_1$ space if and only if each finite subset of X is $\wp CS$.

Theorem 5.8

In $\wp TS$ X, the following conditions are equivalent

(I) X is $\wp T_1$ space

(II) For every $a \in X$, $\{a\}$ is $\wp CS$ in X

(III) Every subset of X is the intersection of all $\wp OSs$ containing it

(IV) The intersection of all $\wp OSs$ containing the point $a \in X$ is $\{a\}$

Proof.

(I) $\Longrightarrow$ (II). Suppose X is a $\wp T_1$ space and $a, b \in X$ with $a \neq b$. Then there exist $\wp OSs$ O_a and O_b such that $a \in O_a$, $b \notin O_a$ and $b \in O_b$, $a \notin O_b$. By the latter condition it then follows that

$$b \in O_b \subseteq X\backslash\{a\} \Longrightarrow X\backslash\{a\} = \cup \{O_b : b \in X\backslash\{a\}\}$$

and so $X\backslash\{a\}$ is $\wp OS$. Consequently $\{a\}$ is $\wp CS$ in X. Hence (II) is true.

(II) $\Longrightarrow$ (III). Suppose $S \subseteq X$ and $a \notin S$. Then $S \subseteq X\backslash \{a\}$ and $X\backslash\{a\}$ is $\wp OS$ in X. We may write $S = \cap \{X\backslash\{a\}: \ a \in X\backslash S\}$. Hence (III) is true.

(III) $\Longrightarrow$ (IV). Obvious.

(IV) $\Longrightarrow$ (I). Suppose a and b are two elements in X such that $a \neq b$. Then there exists a $\wp OS$ O_a containing a but not b and a $\wp OS$ O_b containing b but not a. Hence (I) is true.

Theorem 5.9

A $\wp TS$ X is a $\wp T_2$ space if and only if for any two distinct points a, b in X there exist $\wp CSs$ C_1 and C_2 such that

$$a \in C_1, \ b \notin C_1 \text{ and } b \in C_2, a \notin C_2 \text{ and } X = C_1 \cup C_2.$$

Proof

Suppose X is a $\wp T_2$ space, then for any two distinct points a, b in X, there exist $\wp OSs$ U and V in X such that $a \in U, \ b \in V$ and $U \cap V = \phi$. Set $C_1 = X\backslash V$ and $C_2 = X\backslash U$, then C_1 and C_2 are $\wp CSs$ with $a \in C_1$, $b \notin C_1$ and $y \in C_2$, $a \notin C_2$ and $X = C_1 \cup C_2$.

Conversely, suppose for any a, b in X with $a \neq b$, there exist $\wp CSs$ C_1 and C_2 such that

$$a \in C_1, \ b \notin C_1 \text{ and } b \in C_2, a \notin C_2 \text{ and } X = C_1 \cup C_2.$$

Set $O_1 = X\backslash C_2$ and $O_2 = X\backslash C_1$, then O_1 and O_2 are $\wp OSs$ such that

$$a \in O_1, b \in O_2 \text{ and } O_1 \cap O_2 = \phi$$

Hence X is a $\wp T_2$ space.

Theorem 5.10

A $\wp TS$ X is a $\wp$regular space if and only if for any $\wp OS$ U containing a in X, there exist a $\wp OS$ V containing a such that $\wp Cl(V) \subseteq U$.

Proof. Suppose X is $\wp$regular. Let U be a $\wp OS$ containing $a \in X$, then $X\backslash U$ is a $\wp CS$ in X and $a \notin X\backslash U$. By P-regularity of X, there exist $\wp OSs$ V and W in X such that

$$a \in V, X\backslash U \subseteq W \text{ and } V \cap W = \phi.$$

Notice that $X\backslash W \subseteq U$ and $V \subseteq X\backslash W$. Since $X\backslash W$ is $\wp CS$, so

$$\wp Cl(V) \subseteq \wp Cl(X\backslash W) = X\backslash W \subseteq U.$$

Conversely, let $a \in X$ and C be a $\wp CS$ in X with $a \notin C$, then $a \in X\backslash C$ where $X\backslash C$ is $\wp OS$ in X. By hypothesis, there exists a $\wp OS$ V containing a such that

$\wp Cl\,(V) \subseteq X \backslash C$ implies $C \subseteq X \backslash \wp Cl\,(V)$ where $X \backslash \wp Cl\,(V)$ is a $\wp OS$.

But $V \cap (X \backslash \wp Cl(V)) = V \cap X \backslash V \cap \wp Cl(V) = V \backslash V = \phi$.

Hence X is $\wp$regular.

Theorem 5.11

A $\wp T_0$ and $\wp$regular space is a $\wp T_2$ space.

Proof.

Suppose X is a $\wp T_0$ space which is also $\wp$regular. Let $a, b \in X$ with $a \neq b$. Then there exists a $\wp OS$ U such that $a \in U$ and $b \notin U$. Since U is $\wp OS$ so $X \backslash U$ is a $\wp CS$ and $a \notin X \backslash U$. Also X is $\wp$regular, so there exist $\wp OSs$ V, W and $V \cap W = \phi$ such that $a \in V$ and $X \backslash U \subseteq W$, hence $a \in V$ and $b \in W$. Consequently, X is $\wp T_2$ space.

Conclusion

Separation axioms also known as separation properties have been actively persuaded by many researchers recently in classical as well as bi, tri and quad topological spaces using certain types of open and closed sets. In this work, the separation axioms $\wp T_0$, $\wp T_1$, $\wp T_2$, $\wp T_3$, $\wp$regular have been studied in $\wp TSs$ via $\wp OSs$ and $\wp CSs$ introduced by Khan and Khan (2018). Critical review of previous work concerning separation axioms in the classical, bi, tri, quad and $\wp TSs$ leads to the conclusion that these separation axioms can conveniently be carried over to hexa topological space (a non-void set with six topologies) and even the more general case of n topological space, that is a non-void set equipped with n topologies where n is a natural number.

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