Common Fixed-Point Results for Two Weakly Compatible Expansive Self-Mappings in Cone B-Metric Spaces

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Abstract

In this paper we establish common fixed-point results for two weakly compatible self-mappings in cone b-metric spaces using Expansive mappings. Our results extend, unify and complement various known results existing in the literature.

Keywords: *cone b-metric spaces, common fixed point, expansive self-mappings and weakly compatible mappings*

1. Introduction and Preliminaries

Fixed point theory plays a basic role in many branches of mathematics. There are many works about the fixed point of contractive maps (see [1, 2, 4, 15 and 17]). In 1922, Polish Mathematician Banach [4] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. The concept of b-metric space appeared in some works, such as I.A. Bakhtin [6], S. Czerwik [9], etc. Several papers deal with the fixed-point theory for single valued and multivalued operators in b-metric spaces (see [3,5,6, and 9]). In [6], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalizes the famous Banach contraction principle in metric spaces. In [11] Huang and Zhang introduced cone metric spaces as a generalization of metric spaces. Later on, Kadelburg et.al. [16], obtained a few similar results without normality of the underlying cone but only in the case of quasi-contractive constant $K \in \left[0, \frac{1}{s}\right)$.

In [10], Hussain and Shah introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They established some topological properties in such spaces and improved some recent

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results about KKM mappings in the setting of cone b-metric spaces. The works about cone b-metric spaces and fixed-point theorems for expansive mappings are given by many authors (see [7,8,10,12,13,15 and 20]). In this paper, we present common fixed-point theorems for two weakly compatible expansive self-mappings in cone b-metric spaces. The results greatly generalize and improve the work of [19].

The following definitions and results will be needed in sequel.

Definition 1.1 ***[13].*** Let E be a real Banach space and P be a subset of E. The subset P is called a cone if and only if:

i. P is nonempty, closed and $P \neq \{0\}$

ii. $a,b \in R, \quad a,b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P.$

iii. $P \cap (-P) = \{0\}$

On this basis, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$. Write $\|.\|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$, for all $x, y \in E$.

The least positive number K satisfying the above condition is called the normal constant of P.

Definition 1.2 ***[12].*** Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $0 \leq d(x,y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.3 ***[11] .*** Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $0 \leq d(x,y)$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$

$(ii)\ d(x,y)=d(y,x)$

$(iii)\ d(x,y)\le s\left[d(x,z)+d(z,y)\right]$

The pair $(X,\ d)$ is called a cone b-metric space.

Definition 1.4*[18]*. The mappings $f,g: X\to X$ are weakly compatible if for every $x\in X$, $fgx=gfx$ holds whenever $fx=gx$.

Definition 1.5 *[2]*. Let f and g be self- maps on a set X .

$If\ \ w=fx=gx$ for some $x\in X$,then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g .

Remarks 1.6. The class of cone b-metric spaces is larger than the class of cone metric spaces. Since any cone metric space must be a cone b- metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

Example1.7*[31]*. Let $X=[1,2,3,4]$,

$E=R^2, P=\{(x,y)\ \in E: x\ \ge\ 0\ ,\ y\ge 0\}.$

$Defined\ \ d: X\times X\to E\ \ by$

$$d(x,y)=\begin{cases}\left(\ \mathrm{I}x-yI^{-1}, Ix-y\mathrm{I}^{-1}\right), & if\ \ x\ne y\\ 0, & if\ \ x=y\end{cases}$$

Then (X,d) is a cone b-metric spaces with the coefficient $s=\frac{5}{6}$. But it is not a cone metric space since the triangle inequality is not satisfied. Indeed

$d(1,2)>\ d(1,4)+d(4,2),\ \ d(3,4)>d(3,1)+d(1,4)$

Definition 1.8[10]. Let (X,d) be a cone b-metric space, $x\ \in X$ and $\{x_n\}$ a sequence in X then:

i. $\{x_n\}$ converges to x whenever , for every $c\in E$ with $0<<\ c$, there is a natural number N such that $d(x_n,\ x)\ll$ c for all $n\ge N$ we denote this by $\lim x_n=x$ or $x_n\to x(as\ n\to\infty).$

ii. $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c$ *for all* $n, m \geq N$.

iii. (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent?

The following lemmas are often used (in particular when dealing with cone metric in which the cone need not be normal).

Lemma1.10[10]. Let (X, d) be a cone b-metric space the following properties are often used while dealing with cone b-metric spaces in which the cone is not necessarily normal.

a) *If* $u << v$ *and* $v << w$, *then* $u << w$

b) *If* $0 \leq u << c$ *for each* $c \in \text{int}(P)$, *then* $u = 0$

c) *If* $a \leq b + c$ *for each* $c \in \text{int}(P)$, *then* $a \leq b$

Lemma1.11 [2]. Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$ then w is the unique common fixed point of f and g.

Theorem1.2[19]. Let (X, d) be a cone b-metric spaces with the coefficient $S \geq 1$ and let $a_i \geq 0 (i = 1, 2, 3, 4, 5)$ be constant with $2Sa_1 + (S+1)(a_2 + a_3) + (S^2 + S)(a_4 + a_5) < 2.$

Suppose that the mappings $f, g : X \to X$ satisfy the condition for all $x, y \in X$

$$d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 (gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx)$$

if the range of g contains the range of f and $g(x)$ or $f(x)$ is a complete subspace of X then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

2. Main Results

In this section, we give some common fixed-point results for two weakly compatible expansive self mappings satisfying the contractive condition. In the case of contractive constant $k \in \left[0, \frac{1}{S}\right)$ in cone b-metric spaces without the assumption of normality.

Theorem 2.1. Let (X, d) be a cone b-metric space with co-efficient $S \geq 1$ and suppose $a_i \geq 0$ $(i = 1,2,3)$ be constant with $2Sa_1 + S^2(a_2 + a_3) < 2$. Suppose the mapping $T, f : X \to X$ satisfy the condition for all $x, y \in X$

$$d(Tx, Ty) \geq a_1 d(fx, Tx) + a_2 d(fy, Ty) + a_3 d(fx, Ty) \tag{2.1}$$

If the range of f contains the range of T and $f(x)$ or $T(x)$ is a complete subspace of X then f and T have unique point of coincidence in X.

Moreover if f and T are weakly compatible then T and f have unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq f(X)$ then there exists $x_1 \in X$ such that $fx_0 = Tx_1$. Similarly $x_2 \in X$ such that $fx_1 = Tx_2$ for all $n \in N$, we take a sequence $\{x_n\}$ such that

$fx_n = Tx_{n+1}$ $(for\ all\ n \in N)$. Suppose that, $fx_{n+1} \neq fx_n$ then put $x = x_{n+1},\ y = x_n$ in (2.1), we have

$$d(fx_n, fx_{n-1}) = d(Tx_{n+1}, Tx_n) \geq a_1 d(fx_{n+1}, Tx_{n+1}) + a_2 d(fx_n, Tx_n) + a_3 d(fx_{n+1}, Tx_n)$$

$$d(fx_n, fx_{n-1}) \geq a_1 d(fx_{n+1}, fx_n) + a_2 d(fx_n, fx_{n-1}) + a_3 d(fx_{n+1}, fx_{n-1})$$

$$d(fx_n, fx_{n-1}) \geq a_1 d(fx_{n+1}, fx_n) + a_2 d(fx_n, fx_{n-1}) + Sa_3 d(fx_{n+1}, fx_n) + Sa_3 d(fx_n, fx_{n-1})$$

$$(1 - a_2 - Sa_3) d(fx_n, fx_{n-1}) \geq (a_1 + Sa_3) d(fx_{n+1}, fx_n)$$

$$(a_1 + Sa_3)\ d(fx_{n+1}, fx_n) \leq (1 - a_2 - Sa_3) d(fx_n, fx_{n-1}) \tag{2.2}$$

Now put $x = x_n$ and $y = x_{n+1}$ in (2.1)

$$d(Tx_n, Tx_{n+1}) \geq a_1 d(fx_n, Tx_n) + a_2 d(fx_{n+1}, Tx_{n+1}) + a_3 d(fx_n, Tx_{n+1})$$

$$d(fx_{n-1}, fx_n) \geq a_1 d(fx_n, fx_{n-1}) + a_2 d(fx_{n+1}, fx_n) + a_3 d(fx_n, fx_n)$$

$$a_2 d(fx_{n+1}, fx_n) \leq (1 - a_1) d(fx_n, fx_{n-1}) \tag{2.3}$$

Adding (2.2) and (2.3), we have

$$\left(a_1+a_2+Sa_3\right)d\left(fx_{n+1},\, fx_n\right)\le\left(2-a_1-a_2-Sa_3\right)d\left(fx_n,\, fx_{n-1}\right)$$

$$d\left(fx_{n+1},\, fx_n\right)\le\frac{\left(2-a_1-a_2-Sa_3\right)}{(a_1+a_2+Sa_3)}d\left(fx_n,\, fx_{n-1}\right)$$

Since $2Sa_1+S^2\left(a_2+a_3\right)<2$, we have

$d\left(fx_n,\, fx_{n+1}\right)\le K\ d\left(fx_n,\, fx_{n-1}\right)$ where $K=\dfrac{\left(2-a_1-a_2-Sa_3\right)}{(a_1+a_2+Sa_3)}$ and $K\in\left[0,\dfrac{1}{S}\right)$

$$d\left(fx_n,\, fx_{n+1}\right)\le K\ d\left(fx_n,\, fx_{n-1}\right)\le K^2d\left(fx_{n-1},\, fx_{n-2}\right)\le K^3d\left(fx_{n-2},\, fx_{n-3}\right)\le\cdots\le K^nd\left(fx_{n1},\, fx_0\right)$$

taking any positive integer m and n we have,

$$d\left(fx_n,\, fx_{n+m}\right)\le Sd\left(fx_n,\, fx_{n+1}\right)+Sd\left(fx_{n+1},\, fx_{n+m}\right)$$

$$\le Sd\left(fx_n,fx_{n+1}\right)+S^2d\left(fx_{n+1},fx_{n+2}\right)+S^2d\left(fx_{n+2},fx_{n+m}\right)$$

$$\le Sd\left(fx_n,\, fx_{n+1}\right)+S^2d\left(fx_{n+1},\, fx_{n+2}\right)+S^3d\left(fx_{n+2},\, fx_{n+3}\right)+\ldots..+S^{m-1}d\left(fx_{n+m-2},\, fx_{n+m-1}\right)+S^md\left(fx_{n+m-1},\, fx_{n+m}\right)$$

$$\le\left(Sk^n+S^2k^{n+1}+S^3k^{n+2}+\ldots..+S^mk^{n+m-1}\right)d\left(fx_{1\,,}\, fx_0\right)$$

$$=(SK^n/1-SK)d\left(fx_{1,}\, fx_0\right)$$

Since $K\in\left[0,\dfrac{1}{S}\right)$, we notice that

$$\left(SK^n/1-SK\right)d\left(fx_1,fx_0\right)\to 0\ as\ n\to\infty$$

For any $m\in N$ by Lemma (1.10) for any $c\in Int\left(P\right)$ we can choose $n_0\in N$ such that

$\left(SK^n/1-SK\right)d\left(fx_1,\, fx_0\right)<<c$ for all $n>n_0$

Thus for each $c\in Int\left(P\right)$

$d(fx_{n+m},\, fx_n)\ll c$ for all $n>n_0, m\ge 1$

Therefore $\{fx_n\}$ is a Cauchy sequence in $f(X)$. If $f(X) \subseteq X$ is complete, there exists $q \in p(x)$ and $p \in X$ such that $fx_n \to q$ as $n \to \infty$ and so $fp = q.$ We have to show that $Tp = q$

Put $x = x_{n+1}, y = p$ in (1.1) we have

$$d(fx_n, Tp) = d(Tx_{n+1}, Tp) \geq a_1 d(fx_{n+1}, Tx_{n+1}) + a_2(fp, Tp) + a_3(fx_{n+1}, Tp)$$

$$d(fx_n, Tp) \geq a_1 d(fx_{n+1}, fx_n) + a_2 d(fp, Tp) + Sa_3 d(fx_{n+1}, fx_n) + Sa_3 d(fx_n, Tp)$$

$$(1 - Sa_3) d(fx_n, Tp) \geq (a_1 + Sa_3) d(fx_n, fx_{n+1}) + a_2 d(q, Tp)$$

(2.4)

Also put $x = p, \; y = x_{n+1}$ in (2.1), we have

$$d(Tp, fx_n) = d(Tp, Tx_{n+1}) \geq a_1 d(fp, Tp) + a_2(fx_{n+1}, Tx_{n+1}) + a_3(fp, Tx_{n+1})$$

$$d(Tp, fx_n) \geq a_1 d(fp, Tp) + a_2 d(fx_{n+1,} fx_n) + a_3 d(fp, fx_n)$$

$$d(Tp, fx_n) \geq a_1 d(fp, Tp) + a_2 d(fx_{n+1}, fx_n) + Sa_3 d(q, Tp) + Sa_3 d(Tp, fx_n)$$

$$(1 - Sa_3) d(Tp, fx_n) \geq (a_1 + Sa_3) d(q, Tp) + a_2 d(fx_n, fx_{n+1}) \qquad (2.5)$$

Adding (2.4) & (2.5)

$$(2 - 2Sa_3) d(fx_n, Tp) \geq (a_1 + a_2 + a_3) d(fx_n, fx_{n+1}) + (a_1 + a_2 + Sa_3) d(q, Tp)$$

$$d(fx_n, Tp) \geq \frac{(a_1 + a_2 + Sa_3)}{(2 - 2Sa_3)} d(fx_n, fx_{n+1}) + \frac{(a_1 + a_2 + Sa_3)}{(2 - 2Sa_3)} d(q, Tp)$$

Since $c \in Int(P)$, suppose that

$$\frac{(a_1 + a_2 + Sa_3)}{(2 - 2Sa_3)} d(fx_n, fx_{n+1}) << c \text{ and } \frac{(a_1 + a_2 + Sa_3)}{(2 - 2Sa_3)} d(q, Tp) << c$$

We have

$d(fx_n, Tp) \geq \frac{c}{2} + \frac{c}{2} = c.$ This implies that $d(fx_n, Tp) = 0.$ Hence $fp = Tp = q.$

Assume that there exists u, w in X such that $fu = Tu = w$

$$d(fu, fp) = d(Tu, Tp) \geq a_1 d(fu, Tu) + a_2 d(fp, Tp) + a_3 d(fu, Tp)$$

$$d(fu, fp) \geq a_1 d(w, w) + a_2 d(fp, fp) + a_3 d(fu, fp)$$

$a_3 d(fu, fp) \le d(fu, fp) \Rightarrow (a_3 - 1) d(fu, fp) = 0$ this implies that, $d(fu, fp) = 0$. Thus $fu = fp$ and so $fu = fp = q$.

Moreover, the mappings T and f are weakly compatible by lemma (1.11) we know that q is the unique common fixed point of f and T.

Theorem 2.2. Let (X, d) be a complete cone b-metric space, with $S \ge 1$ and let $a_i \ge 0$ $(i = 1, 2, 3, 4)$ be constant with $2Sa_1 + S^2(a_2 + a_3) + a_4 < 2$ and the mappings $T, f : X \to X$ are the continuous onto and satisfies the contractive condition, for all $x, y \in X$

$$d(Tx, Ty) \ge a_1 d(fx, fy) + a_2 d(fx, Ty) + a_3 (Tx, fy) + a_4 (fx, Tx)$$
(2.6)

If the range of f contains the range of T and $f(X)$ or $T(X)$ is complete subspace of X then f and T have a unique point of coincidence in X .

Moreover if f and T are weakly compatible, then T and f have unique common fixed point in X .

Proof. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq f(X)$ then there exists $x_1 \in X$ such that $fx_0 = Tx_1$. By induction a sequence $\{x_n\}$ can be chosen such that $fx_n = Tx_{n+1}$ for all $n \in N$

Suppose that $fx_{n+1} \ne fx_n$ then put $x = x_{n+1}$, $y = x_n$ in (2.6) we have

$$d(fx_n, fx_{n-1}) = d(Tx_{n+1}, Tx_n)$$

$$\ge a_1 d(fx_{n+1}, fx_n) + a_2 d(fx_{n+1}, Tx_n) + a_3 d(Tx_{n+1}, fx_n) + a_4 d(fx_{n+1}, Tx_{n+1})$$

$$d(fx_n, fx_{n-1}) \ge a_1(fx_{n+1}, fx_n) + a_2 d(fx_{n+1}, fx_{n-1}) + a_3 d(fx_n, fx_n) + a_4 d(fx_{n+1}, fx_n)$$

$$d(fx_n, fx_{n-1}) \ge a_1(fx_{n+1}, fx_n) + Sa_2 d(fx_{n+1}, fx_n) + Sa_2(fx_n, fx_{n-1}) + a_4 d(fx_{n+1}, fx_n)$$

$$(a_1 + Sa_2 + a_4) d(fx_{n+1}, fx_n) \le (1 - Sa_2) d(fx_n, fx_{n-1}) \tag{2.7}$$

Also put $x = x_n$, $y = x_{n+1}$ in (2.6) we have

$$d(fx_{n-1}, fx_n) = d(Tx_n, Tx_{n+1})$$

$$\ge a_1 d(fx_n, fx_{n+1}) + a_2 d(fx_n, Tx_{n+1}) + a_3 d(Tx_n, fx_{n+1}) + a_4 d(fx_n, Tx_n)$$

$$d\left(fx_{n-1},fx_{n}\right)\geq a_1 d\left(fx_{n},fx_{n+1}\right)+a_2 d\left(fx_{n},fx_{n}\right)+a_3 d\left(fx_{n-1},fx_{n+1}\right)+a_4 d\left(fx_{n},fx_{n-1}\right)$$

$$d\left(fx_{n-1},fx_{n}\right)\geq a_1 d\left(fx_{n},fx_{n+1}\right)+Sa_3 d\left(fx_{n-1},fx_{n}\right)+Sa_3 d\left(fx_{n},fx_{n+1}\right)+a_4 d\left(fx_{n},fx_{n-1}\right)$$

$$\left(1-Sa_3-a_4\right)d\left(fx_{n-1},fx_{n}\right)\geq\left(a_1+Sa_3\right)d\left(fx_{n},fx_{n+1}\right)$$

$$\left(a_1+Sa_3\right)d\left(fx_{n},fx_{n+1}\right)\leq\left(1-Sa_3-a_4\right)d\left(fx_{n-1},fx_{n}\right) \tag{2.8}$$

Adding (2.7) & (2.8) we have

$$\left(2a_1+Sa_2+Sa_3+a_4\right)d\left(fx_{n},fx_{n+1}\right)\leq\left(2-Sa_2-Sa_3-a_4\right)d\left(fx_{n-1},fx_{n}\right)$$

$$d\left(fx_{n+1},fx_{n}\right)\leq\frac{2-Sa_2-Sa_3-a_4}{2a_1+Sa_2+Sa_3+a_4}d\left(fx_{n},fx_{n-1}\right)$$

Since $2Sa_1+S^2\left(a_2+a_3\right)+a_4<2,$ so we take

$$K=\frac{2-Sa_2-Sa_3-a_4}{2a_1+Sa_2+Sa_3+a_4} \text{ and } K=\left[0,\frac{1}{S}\right)$$

$$d\left(fx_{n},fx_{n+1}\right)\leq Kd\left(fx_{n},fx_{n-1}\right)\leq K^2 d\left(fx_{n-1},fx_{n-2}\right)\leq K^3 d\left(fx_{n-2},fx_{n-3}\right)\leq\ldots\leq K^n d\left(fx_{n1},fx_{0}\right)$$

taking any positive integer m and n we have,

$$d\left(fx_{n},fx_{n+m}\right)\leq Sd\left(fx_{n},fx_{n+1}\right)+Sd\left(fx_{n+1},fx_{n+m}\right)$$

$$\leq Sd\left(fx_{n},fx_{n+1}\right)+S^2 d\left(fx_{n+1},fx_{n+2}\right)+S^2 d\left(fx_{n+2},fx_{n+m}\right)$$

$$\leq Sd\left(fx_{n},fx_{n+1}\right)+S^2 d\left(fx_{n+1},fx_{n+2}\right)+S^3 d\left(fx_{n+2},fx_{n+3}\right)+\ldots+S^{m-1}d\left(fx_{n+m-2},fx_{n+m-1}\right)+S^m d\left(fx_{n+m-1},fx_{n+m}\right)$$

$$\leq\left(Sk^n+S^2k^{n+1}+S^3k^{n+2}+\ldots..+S^m k^{n+m-1}\right)d\left(fx_{1},fx_{0}\right)$$

$$=SK^n/1-SK)d\left(fx_{1},fx_{0}\right)$$

Since $K=\left[0,\frac{1}{S}\right)$, we notice that

$$\left(SK^n/1-SK\right)d\left(fx_1,fx_0\right)\to 0 \text{ as } n\to\infty$$

For any $m\in\mathrm{N}$ by lemma (1.10) for any $c\in Int\left(P\right)$ we can choose $n_0\in\mathrm{N}$ such that

$$\left(SK^n/1-SK\right)d\left(fx_1,fx_0\right)<<c \text{ for all } n>n_0$$

Thus, for each $\mathrm{c}\in Int\left(\mathrm{P}\right)$

$d(fx_{n+m}, fx_n) << c$ for all $n > n_0, m \geq 1.$ Therefore $\{fx_n\}$ is a Cauchy sequence in $f(X)$. If $f(X) \subseteq X$ is complete, there exists $q \in p(x)$ and $p \in X$ such that $fx_n \to q$ as $n \to \infty$ and so $fp = q$. We have to show that $Tp = q$.

Put $x = x_{n+1}, y = p$ in (2.6) we have

$$d(fx_n, Tp) = d(Tx_{n+1}, Tp)$$
$$\geq a_1 d(fx_{n+1}, fp) + a_2 d(fx_{n+1}, Tp) + a_3 d(Tx_{n+1}, fp) + a_4 d(fx_{n+1}, Tx_{n+1})$$
$$\geq a_1 d(fx_{n+1}, fp) + a_2 d(fx_{n+1}, Tp) + a_3 d(fx_n, fp) + a_4 d(fx_{n+1}, fx_n)$$
$$d(fx_n, Tp) \geq a_1 d(fx_{n+1}, q) + a_2 d(fx_{n+1}, Tp) + a_3 d(fx_n, q) + a_4 d(fx_{n+1}, fx_n) \tag{2.9}$$

Also put $x = p$ and $y = x_{n+1}$ in (2.6) we have

$$d(Tp, fx_n) = d(Tp, Tx_{n+1}) \geq a_1 d(fp, fx_{n+1}) + a_2 d(fp, Tx_{n+1}) + a_3 d(Tp, fx_{n+1}) + a_4 d(fp, Tp)$$
(2.10)

Adding (2.9) & (2.10) we have

$$2d(fx_n, Tp) \geq 2a_1 d(fx_{n+1}, q) + (a_2 + a_3) d(Tp, fx_{n+1}) + (a_2 + a_3) d(fx_n, q) + a_4 d(fx_n, fx_{n+1}) + a_4 d(q, Tp)$$
$$2d(fx_n, Tp) \geq 2Sa_1 d(fx_{n+1} fx_n) + 2Sa_1 d(fx_n, q) + S(a_2 + a_3) d(Tp, fx_n) + S(a_2 + a_3) d(fx_n, fx_{n+1})$$
$$+(a_2 + a_3) d(fx, fx_{n+1}) + Sa_4 d(q, fx_n) + Sa_4 d(fx_n, Tp)$$
$$(2 - Sa_2 - Sa_3 - Sa_4) d(fx_n, Tp) \geq (2Sa_1 + Sa_{2+} Sa_3 + a_4) d(fx_n, fx_{n+1}) + (2Sa_1 + a_2 + a_3 + Sa_4) d(q, fx_n)$$
$$d(fx_n, Tp) \geq \frac{(2Sa_1 + Sa_2 + Sa_3 + a_4)}{(2 - Sa_2 - Sa_3 - Sa_4)} d(fx_n, Tp) + \frac{(2Sa_1 + a_2 + a_3 + Sa_4)}{(2 - Sa_2 - Sa_3 - Sa_4)} d(q, fx_n)$$

Since $c \in Int(P)$ suppose that

$$\frac{(2Sa_1 + Sa_2 + Sa_3 + a_4)}{(2 - Sa_2 - Sa_3 - Sa_4)} d(fx_n, Tp) << \frac{c}{2}$$ and

$$\frac{(2Sa_1 + a_2 + a_3 + Sa_4)}{(2 - Sa_2 - Sa_3 - Sa_4)} d(q, fx_n) << \frac{c}{2}$$ then

$d(fx_n, Tp) \geq \frac{c}{2} + \frac{c}{2} = c.$ This implies that $d(fx_n, Tp) = 0.$ Hence $fp = q = Tp.$

Assume that there exists u, w in X such that $fu = Tu = w.$

$d(fu, fp) = d(Tu, Tp) \geq a_1 d(fu, fp) + a_2 d(fu, Tp) + a_3 d(Tu, fp) + a_4 d(fp, Tp)$

$d(fu, fp) \geq a_1 d(fu, fp) + a_2 d(fu, fp) + a_3 d(fu, fp) + a_4 d(fp, fp)$

$(1 - a_{1\_} a_2 - a_3) d(fu, fp) \geq 0$. This implies that $d(fu, fp) = 0$. Hence $w = fu = fp = q.$

Moreover the mappings T and f are weakly compatible, by lemma (1.11) we know that q is the unique common fixed point of f and T .

Theorem 2.3. Let (X, d) be a complete cone b-metric space, with $S \geq 1$ and let $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ be constant with $2Sa_1 + S^2(a_2 + a_3) + a_4 + a_5 < 2$ and the mappings $T, f : X \to X$ are the continuous onto and satisfies the contractive condition, for all $x, y \in X$

$$d(Tx, Ty) \geq a_1 d(fx, Ty) + a_2 d(Tx, fy) + a_3 d(fx, Tx) + a_4 d(fx, fy) + a_5 d(fy, Ty) \quad (2.11)$$

If the range of f contains the range of T and $f(X)$ or $T(X)$ is complete subspace of X then f and T have a unique point of coincidence in X .

Moreover if f and T are weakly compatible, then T and f have unique common fixed point in X .

Proof: Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq f(X)$ then there exists $x_1 \in X$ such that $fx_0 = Tx_1$. By induction a sequence $\{x_n\}$ can be chosen such that $fx_n = Tx_{n+1}$ for all $n \in \mathrm{N}$

Suppose that $fx_{n+1} \neq fx_n$ then put $x = x_{n+1}, \ y = x_n$ in (2.11) we have

$d(fx_n, fx_{n-1}) = d(Tx_{n+1}, Tx_n)$

$\geq a_1 d(fx_{n+1}, Tx_n) + a_2 d(Tx_{n+1}, fx_n) + a_3 d(fx_{n+1}, Tx_{n+1}) + a_4 d(fx_{n+1}, fx_n) + a_5 d(fx_n, Tx_n)$

$\geq a_1(fx_{n+1}, fx_{n-1}) + a_2 d(fx_n, fx_n) + a_3 d(fx_{n+1}, fx_n) + a_4 d(fx_{n+1}, fx_n) + a_5 d(fx_n, fx_{n-1})$

$d(fx_n, fx_{n-1}) \geq Sa_1(fx_{n+1}, fx_n) + Sa_1 d(fx_n, fx_{n-1}) + a_3 d(fx_{n+1}, fx_n)) + a_4 d(fx_{n+1}, fx_n) + a_5 d(fx_n, fx_{n-1})$

$(1 - Sa_1 - a_5) d(fx_n, fx_{n-1}) \geq (Sa_1 + a_3 + a_4) d(fx_{n+1}, fx_n)$

$$(Sa_1 + a_3 + a_4) d(fx_{n+1}, fx_n) \leq (1 - Sa_1 - a_5) d(fx_n, fx_{n-1}) \quad (2.12)$$

Also put $x = x_n$, $y = x_{n+1}$ in (2.11) we have

$$d\left(fx_{n-1,}fx_n\right) = d\left(Tx_n, Tx_{n+1}\right)$$

$$d\left(fx_{n-1}, fx_n\right) \geq a_1 d\left(fx_n, Tx_{n+1}\right) + a_2 d\left(Tx_n, fx_{n+1}\right) + a_3 d\left(fx_n, Tx_n\right) + a_4 d\left(fx_{n,}fx_{n+1}\right) + a_5 d\left(fx_{n+1}, Tx_{n+1}\right)$$

$$d\left(fx_{n-1}, fx_n\right) \geq a_1 d\left(fx_n, fx_n\right) + a_2 d\left(fx_{n-1}, fx_{n+1}\right) + a_3 d\left(fx_n, fx_{n-1}\right) + a_4 d\left(fx_{n,}fx_{n+1}\right) + a_5 d\left(fx_{n+1}, fx_n\right)$$

$$\left(1 - Sa_2 - a_3\right) d\left(fx_{n-1}, fx_n\right) \geq \left(Sa_2 + a_4 + a_5\right) d\left(fx_n, fx_{n+1}\right)$$

$$\left(Sa_2 + a_4 + a_5\right) d\left(fx_n, fx_{n+1}\right) \leq \left(1 - Sa_2 - a_3\right) d\left(fx_{n-1}, fx_n\right) \qquad (2.13)$$

Adding (2.12) & (2.13) we have

$$\left(Sa_1 + Sa_{2+}a_3 + 2a_4 + a_5\right) d\left(fx_n, fx_{n+1}\right) \leq \left(2 - Sa_1 - Sa_2 - a_3 - a_5\right) d\left(fx_{n-1}, fx_n\right)$$

$$d\left(fx_{n+1}, fx_n\right) \leq \frac{\left(2 - Sa_1 - Sa_2 - a_3 - a_5\right)}{\left(Sa_1 + Sa_2 + a_3 + 2a_4 + a_5\right)} d\left(fx_n, fx_{n-1}\right)$$

Since $2Sa_1 + S^2\left(a_2 + a_3\right) + a_4 + a_5 < 2$, so we take

$$K = \frac{\left(2 - Sa_1 - Sa_2 - a_3 - a_5\right)}{\left(Sa_1 + Sa_2 + a_3 + 2a_4 + a_5\right)} \text{ and } K = \left[0, \frac{1}{S}\right)$$

$$d\left(fx_n, fx_{n+1}\right) \leq Kd\left(fx_n, fx_{n-1}\right) \leq K^2 d\left(fx_{n-1}, fx_{n-2}\right) \leq K^3 d\left(fx_{n-2}, fx_{n-3}\right) \leq \ldots \leq K^n d\left(fx_{n1}, fx_0\right)$$

taking any positive integer m & n we have,

$$d\left(fx_n, fx_{n+m}\right) \leq Sd\left(fx_n, fx_{n+1}\right) + Sd\left(fx_{n+1}, fx_{n+m}\right)$$

$$\leq Sd\left(fx_n, fx_{n+1}\right) + S^2 d\left(fx_{n+1}, fx_{n+2}\right) + S^2 d\left(fx_{n+2}, fx_{n+m}\right)$$

$$\leq Sd\left(fx_n, fx_{n+1}\right) + S^2 d\left(fx_{n+1}, fx_{n+2}\right) + S^3 d\left(fx_{n+2}, fx_{n+3}\right) + \ldots + S^{m-1} d\left(fx_{n+m-2}, fx_{n+m-1}\right) + S^m d\left(fx_{n+m-1}, fx_{n+m}\right)$$

$$\leq \left(Sk^n + s^2 k^{n+1} + S^3 k^{n+2} + \ldots.. + S^m k^{n+m-1}\right) d\left(fx_{1\,,}\, fx_0\right)$$

$$= (SK^n / 1 - SK) d\left(fx_{1,}\, fx_0\right)$$

Since $K \in \left[0, \frac{1}{S}\right)$, we notice that

$$\left(SK^n / 1 - SK\right) d\left(fx_1, fx_0\right) \to 0 \text{ as } n \to \infty$$

For any $m \in N$ by lemma (1.10) for any $c \in Int\left(P\right)$ we can choose $n_0 \in N$ such that

$\left(SK^n/1-SK\right)d\left(fx_1, fx_0\right) << c$ for all $n_0 \in N$. Thus, for each $c \in Int\left(P\right)$

$d(fx_{n+m}, fx_n) << c$ for all $n > n_0, m \geq 1$. Therefore $\{fx_n\}$ is a Cauchy sequence in $f\left(X\right)$. If $f\left(X\right) \subseteq X$ is complete, there exists $q \in p\left(x\right)$ and $p \in X$ such that $fx_n \to q$ as $n \to \infty$ and so $fp = q$. We have to show that $Tp = q$

Put $x = x_{n+1}, y = p$ in (2.11) we have

$$d\left(fx_n, Tp\right) = d\left(Tx_{n+1}, Tp\right) \geq a_1 d\left(fx_{n+1}, Tp\right) + a_2 d\left(Tx_{n+1}, fp\right) + a_3 d\left(fx_{n+1}, Tx_{n+1}\right) + a_4 d\left(fx_{n+1}, fp\right) + a_5 d\left(fp, Tp\right)$$

$$\geq a_1 d\left(fx_{n+1}, Tp\right) + a_2 d\left(fx_n, fp\right) + a_3 d\left(fx_{n+1}, fx_n\right) + a_4 d\left(fx_{n+1}, fp\right) + a_5 d\left(fp, Tp\right)$$

$$d\left(fx_n, Tp\right) \geq a_1 d\left(fx_{n+1}, Tp\right) + a_2 d\left(fx_n, q\right) + a_3 d\left(fx_{n+1}, fx_n\right) + a_4 d\left(fx_{n+1}, q\right) + a_5 d\left(q, Tp\right) \qquad (2.14)$$

Also put $x = p$ and $y = x_{n+1}$ in (2.11) we have

$$d\left(Tp, fx_n\right) = d\left(Tp, Tx_{n+1}\right) \geq a_1 d\left(fp, Tx_{n+1}\right) + a_2 d\left(Tp, fx_{n+1}\right) + a_3 d\left(fp, Tp\right) + a_4 d\left(fp, fx_{n+1}\right) + a_5 d\left(fx_{n+1}, Tx_{n+1}\right)$$

$$d\left(T, fx_n\right) \geq a_1 d\left(q, fx_n\right) + a_2 d\left(Tp, fx_{n+1}\right) + a_3 d\left(q, Tp\right) + a_4 d\left(q, fx_{n+1}\right) + a_5 d\left(fx_{n+1}, fx_n\right) \qquad (2.15)$$

Adding (2.14) & (2.15) we have

$$2d\left(fx_n, Tp\right) \geq \left(a_1 + a_2\right) d\left(Tp, fx_{n+1}\right) + \left(a_1 + a_2\right) d\left(fx_n, q\right) + \left(a_3 + a_5\right) d\left(fx_n, fx_{n+1}\right) + 2a_4 d\left(q, fx_{n+1}\right)$$
$$+\left(a_3 + a_5\right) d\left(q, Tp\right)$$

$$2d\left(fx_n, Tp\right) \geq S\left(a_1 + a_2\right) d\left(fx_{n+1} fx_n\right) + S\left(a_1 + a_2\right) d\left(fx_n, Tp\right) + S\left(a_2 + a_3\right) d\left(Tp, fx_n\right) + \left(a_3 + a_5\right) d\left(fx_n, fx_{n+1}\right)$$
$$+\left(a_1 + a_2\right) d\left(fx_n, q\right) + 2Sa_4 d\left(q, fx_n\right) + 2Sa_4 d\left(fx_n, fx_{n+1}\right) + S\left(a_3 + a_5\right) d\left(fx_n, q\right)$$

$$2d\left(fx_n, Tp\right) \geq S\left(a_1 + a_2\right) d\left(fx_{n+1} fx_n\right) + S\left(a_1 + a_2\right) d\left(fx_n, p\right) + S\left(a_2 + a_3\right) d\left(Tp, fx_n\right) + \left(a_3 + a_5\right) d\left(fx_n, fx_{n+1}\right)$$
$$+\left(a_1 + a_2\right) d\left(fx_n, q\right) + 2Sa_4 d\left(q, fx_n\right) + 2Sa_4 d\left(fx_n, fx_{n+1}\right) + S\left(a_3 + a_5\right) d\left(fx_n, q\right)$$

$$\left(2 - Sa_1 - Sa_2 - Sa_3 - Sa_5\right) d\left(fx_n, Tp\right) \geq \left(Sa_1 + Sa_2 + a_3 + 2Sa_4 + a_5\right) d\left(fx_n, fx_{n+1}\right)$$
$$+\left(a_1 + a_2 + Sa_3 + 2Sa_4 + Sa_5\right) d\left(q, fx_n\right)$$

$$d\left(fx_n, Tp\right) \geq \frac{\left(Sa_1 + Sa_2 + a_3 + 2Sa_4 + a_5\right)}{\left(2 - Sa_1 - Sa_2 - Sa_3 - Sa_5\right)} d\left(fx_n, Tp\right) + \frac{\left(a_1 + a_2 + Sa_3 + 2Sa_4 + Sa_5\right)}{\left(2 - Sa_1 - Sa_2 - Sa_3 - Sa_5\right)} d\left(q, fx_n\right)$$

Since $c \in Int\left(p\right)$, suppose that

$$\frac{\left(Sa_1+Sa_2+a_3+2Sa_4+a_5\right)}{\left(2-Sa_1-Sa_2-Sa_3-Sa_5\right)}d\left(fx_n,Tp\right)<<\frac{c}{2},$$

$$\frac{\left(a_1+a_2+Sa_3+2Sa_4+Sa_5\right)}{\left(2-Sa_1-Sa_2-Sa_3-Sa_5\right)}d\left(q,fx_n\right)<<\frac{c}{2}$$

Then $d\left(fx_n,Tp\right)\geq\frac{c}{2}+\frac{c}{2}=c.$ This implies that $d\left(fx_n,Tp\right)=0$.

Hence $fp=q=Tp.$

Assume that there exists u,w in X such that $fu=Tu=w$.

$$d\left(fu,fp\right)=d\left(Tu,Tp\right)\geq a_1d\left(fu,Tp\right)+a_2d\left(Tu,fp\right)+a_3d\left(fu,fu\right)+a_4d\left(fu,fp\right)+a_5d\left(fp,Tp\right)$$

$$d\left(fu,fp\right)\geq a_1d\left(fu,fp\right)+a_2d\left(fu,fp\right)+a_3d\left(fu,fp\right)+a_4d\left(fp,fp\right)+a_5d\left(fp,fp\right)$$

$$d\left(fu,fp\right)\geq(a_1+a_2+a_3)d(fu,fp)$$

Since $0\leq a_1+a_2+a_3<1$ by lemma (1.10) we can obtain that $d\left(fu,fp\right)=0.$ Hence $w=fu=fp=q$. Moreover the mappings T and f are weakly compatible, by lemma (1.11) we know that q is the unique common fixed point of f and T .

Corollary 2.4. If $f=I$ in the theorem 2.2, we get the result 2.1 of [19].

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