Certain Type of Continuity in Penta Topological Spaces

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Abstract

In this article, we aim at to present new sorts of open sets and closed sets viz. ᵽ-Ƅ open set, ᵽ-Ƅ closed set, ᵽ-Ƅτ open set and ᵽ-Ƅτ closed set in the setting of penta topological spaces and investigate some of their properties. In addition, the notion of ᵽ-Ƅ continuity and ᵽ-Ƅ homeomorphism is defined and some related results are proved.

***Keywords*:** ᵽ-Ƅ open set, ᵽ-Ƅ closed set, ᵽ-Ƅτ open set, ᵽ-Ƅτ closed set, ᵽ-Ƅ continuity, ᵽ-Ƅ homeomorphism.

Introduction

In the last few decades the notion of classical topological space has been extended to bi-topological space (BTS), tri-topological space (TTS) and quad-topological space (QTS). The idea of BTS was initiated by Kelly (1963). Subsequent work in the area has been done by Fletcher et al. (1969), Kim (1968), Lane (1967), Patty (1967), Pervin (1967) and Reilly (1972, 1973). Kovar (2000) introduced the concept of TTS which was further investigated by Priyadharsini and Parvathi (2017).

QTS was introduced by Mukundan (2013). Tapi and Sharma (2015) studied Q-B continuous functions in QTSs. As a natural generalization of these concepts, Khan and Khan (2018) introduced the notion of penta-topological space (PTS) and also developed the idea of p-continuity and p-homeomorphism therein. Following the work of Khan and Khan (2018), Anjaline & Pricilla (2020) and Pacifica and Fatima (2019) discussed some new topological concepts in Penta Topological Spaces (PTSs). More recently, Yaseen et al. (2021) discussed some characteristics of penta- open sets in penta topological spaces.

In this paper, ᵽ-Ƅ open set, ᵽ-Ƅ closed set, ᵽ-Ƅτ open set, ᵽ-Ƅτ closed set are defined and some of their properties are investigated. In addition, the conception of ᵽ-Ƅ continuity and ᵽ-Ƅ homeomorphism in PTSs is discussed and some results concerning these ideas are proved.

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Preliminary Results

Here, we mention some elementary concepts and results pertaining to BTS, TTS, QTS and PTS. Exhaustive details about classical topological spaces, may be found in Munkres (2000), Simmons (1963) and Willard (1970).

Definition 2.1

A non-empty Ӿ together with

(a) two topologies τ_1, τ_2 i.e. $(Ӿ, \tau_1, \tau_2)$ is called BTS,
(b) three topologies τ_1, τ_2, τ_3 i.e. $(Ӿ, \tau_1, \tau_2, \tau_3)$ is called TTS,
(c) four topologies $\tau_1, \tau_2, \tau_3, \tau_4$ i.e. $(Ӿ, \tau_1, \tau_2, \tau_3, \tau_4)$ is called QTS,
(d) five topologies $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ i.e. $(Ӿ, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ is called PTS.

Definition 2.2

In BTS $(Ӿ, \tau_1, \tau_2)$, elements of topologies τ_1 and τ_2 are called τ_1*-open* sets and τ_2*-open* sets respectively and their relative complements are termed as τ_1*-closed* sets and τ_2*-closed* sets.

Definition 2.3

Let $(Ӿ, \tau_1, \tau_2, \tau_3)$ be a TTS and $S \subseteq Ӿ$. Then S is called a *tri-open* set in Ӿ if $S \in \tau_1 \cup \tau_2 \cup \tau_3$. A tri-open set in Ӿ is also known as *123-open* in Ӿ.

Definition 2.4

Let $(Ӿ, \tau_1, \tau_2, \tau_3, \tau_4)$ be a QTS and $S \subseteq Ӿ$. Then S is said to be *quad-open (ҩ-open)* set if $S \epsilon \tau_1 \cup \tau_2 \cup \tau_3 \cup \tau_4$ and its complement is termed as *quad-closed (ҩ-closed)* set. The 4-tuple of topologies $(\tau_1, \tau_{2\tau}, \tau_3, \tau_4)$ is called $quad$ (or ҩ) $- topology$.

Notation 2.5

A PTS is denoted by $(Ӿ, \tau_Ᵽ)$ where $\tau_Ᵽ = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ is called *penta (or Ᵽ)- topology* on Ӿ.

Definition 2.6

In PTS $(Ӿ, \tau_Ᵽ)$ elements of τ_i; $i = 1, 2, 3, 4, 5$ are named τ_i*-open* sets and their relative complements are known as τ_i*-closed* sets.

Definition 2.7
Let (Ӿ, $\tau_{ᵽ}$) be a PTS and $S \subseteq$ Ӿ. We call S as *penta-open (ᵽ -open)* if
$S \epsilon \cup \tau_i \,; i = 1, 2, 3, 4, 5$, whereas the complement of S is titled as *penta-closed (ᵽ -closed).*

Definition 2.8
Let (Ӿ, $\tau_{ᵽ}$) be a PTS and $S \subseteq$ Ӿ. Then S is said to be *ᵽ-neighborhood* of a point $a \in$ Ӿ if and only if there exists a ᵽ-open set O such that $a \in O \subseteq S$.

Definition 2.9
If (Ӿ, $\tau_{ᵽ}$) is a PTS and $S \subseteq$ Ӿ, then a point $a \in S$ is called *ᵽ-Interior* point of S if there exists a ᵽ-open set O such that $a \in O \subseteq S$. The set of all p-interior points of S is known as *ᵽ-Interior* of S and is written as ᵽ-Int*(S)*. Being union of all ᵽ-open sets contained in *S*, it follows that ᵽ-Int*(S)* is the largest ᵽ-open set contained in S.

Theorem 2.10
Let (Ӿ, $\tau_{ᵽ}$) be a PTS and $R \subseteq$ Ӿ, $S \subseteq$ Ӿ. Then

1. *R* is ᵽ-open iff *R* = ᵽ-Int(*R*).
2. ᵽ-Int($R \cup S$) ⊒ ᵽ-Int(R) ∪ ᵽ-Int(S).

Definition 2.11
Let (Ӿ, $\tau_{ᵽ}$) be a PTS. The *ᵽ -closure* of a subset *S* of Ӿ, denoted by ᵽ-cl*(S)* is defined as ᵽ-cl*(S)* = $\cap_{\alpha \epsilon I} C_\alpha$, where each $C_\alpha, \alpha \epsilon I$ is a ᵽ-closed set in *X* containing *S.* Simple arguments imply that ᵽ-cl(*S*) is a least ᵽ-closed set with *S* ⊆ ᵽ-cl(*S*).

Theorem 2.12
Let (Ӿ, $\tau_{ᵽ}$) be a PTS and $S \subseteq$ Ӿ. Then

1. *S* is ᵽ-closed iff *S* = ᵽ-cl(*S*).
2. (ᵽ-Int(*S*))^c = ᵽ-cl(*S*)^c.

Certain Open and Closed Sets in Penta Topological Spaces

In this section, we introduce ᵽ-Ƅ open set, ᵽ-Ƅ closed set, ᵽ-Ƅτ open set, ᵽ-Ƅτ closed set in PTS and also discuss certain properties of these sets.

Definition 3.1
A subset S of a topological space Ӿ is called

1. *Ƀ-open* if $S \subseteq \text{Int}(\text{cl}(S)) \cup \text{cl}(\text{Int}(S))$
2. *Ƀ-closed* if $\text{Int}(\text{cl}(S)) \cap \text{cl}(\text{Int}(S)) \subseteq S$

Definition 3.2
Let S be a subset of a PTS (Ӿ, $\tau_{ᵽ}$). Then

1. S is said to be *ᵽ-Ƀ open* set if $S \subseteq$ ᵽ-Int[ᵽ-cl(S)] ∪ ᵽ-cl[ᵽ-Int(S)].
2. S is said to be *ᵽ-Ƀ closed* set if ᵽ-Int[ᵽ-cl(S)] ∩ ᵽ-cl[ᵽ-Int(S)] ⊆ S.
3. the *ᵽ-Ƀ Interior* of S is denoted by ᵽ-Ƀ Int(S) and defined as
   ᵽ-Ƀ Int(S) = $\bigcup_{\lambda \epsilon I}\{O_\lambda : O_\lambda \subseteq S$ where each O_λ is ᵽ-Ƀ open$\}$.
4. the *ᵽ-Ƀ closure* of S is denoted by ᵽ-Ƀ cl(S) and is defined as
   ᵽ-Ƀ cl(S) = $\bigcap_{\lambda \epsilon I}\{C_\lambda : S \subseteq C_\lambda$ where each C_λ is ᵽ-Ƀ closed$\}$.

Theorem 3.3
In PTS (Ӿ, $\tau_{ᵽ}$), an arbitrary union of ᵽ-Ƀ open sets is ᵽ-Ƀ open.

Proof
Let $\{O_\lambda : \lambda \in I\}$ be a family of ᵽ-Ƀ open sets in (Ӿ, $\tau_{ᵽ}$). For each $\lambda \in I$, $O_\lambda \subseteq$ ᵽ-Int[ᵽ-cl(S)] ∪ ᵽ-cl[ᵽ-Int(S)]. Therefore ∪ $O_\lambda \subseteq$ ᵽ-Int[ᵽ-cl(S)] ∪ ᵽ-cl[ᵽ-Int(S)].
Hence ∪ O_λ is ᵽ-Ƀ open.

Remarks 3.4
Notice that ᵽ-Ƀ Int(S) is the largest ᵽ-Ƀ open set with ᵽ-Ƀ Int(S) ⊆ S.

Theorem 3.5
Suppose (Ӿ, $\tau_{ᵽ}$) is a PTS and $S \subseteq$ Ӿ**.** Then S is ᵽ-Ƀ open if and only if S = ᵽ-Ƀ Int(S).

Proof

Suppose that S is a ꝑ-Ƅ open set in Ӿ. For $\lambda \epsilon I$, consider the collection $\mathcal{B} = \{O_\lambda : O_\lambda \subseteq S$ where each O_λ is ꝑ-Ƅ open$\}$. It is clear that $S \epsilon \mathcal{B}$ and each member of $\mathcal{B}$ is a subset of S. It follows that $\cup \mathcal{B} = S$ and hence S = ꝑ-Ƅ Int(S).

Conversely, suppose that S = ꝑ-Ƅ Int(S). Since ꝑ-Ƅ Int(S) is ꝑ-Ƅ open set so is S.

Theorem 3.6

Let (Ӿ, τ_p) be a PTS and $R \subseteq$ Ӿ, $S \subseteq$ Ӿ. Then ꝑ-Ƅ Int(R ∪ S) ⊇ ꝑ-Ƅ Int(R) ∪ ꝑ-Ƅ Int(S).

Proof

We have that ꝑ-Ƅ Int(R) ⊆ R, ꝑ-Ƅ Int(R) is p-open. Likewise ꝑ-Ƅ Int(S) ⊆ S, ꝑ-Ƅ Int(S) is ꝑ-open. Then ꝑ-Ƅ Int(R) ∪ ꝑ-Ƅ Int(S) ⊆ R∪ S. Since ꝑ-Ƅ Int(R) ∪ ꝑ-Ƅ Int(S) is a ꝑ-Ƅ open set in R∪ S and ꝑ-Ƅ Int(R∪ S) is the largest ꝑ-Ƅ open set in R∪ S, we get ꝑ-Ƅ Int(R∪ S) ⊇ ꝑ-Ƅ Int(R) ∪ ꝑ-Ƅ Int(S).

Theorem 3.7

In PTS (Ӿ, $\tau_{ꝑ}$), an arbitrary Intersection of ꝑ-Ƅ closed sets is ꝑ-Ƅ closed.

Proof

Let $\{C_\lambda : \lambda \epsilon I\}$ and $\{O_\lambda : \lambda \epsilon I\}$ be collections of ꝑ-Ƅ closed sets and ꝑ-Ƅ open sets in Ӿ respectively. Take, $C_\lambda = O_\lambda^c$. Since $\cup O_\lambda$ is ꝑ-Ƅ open set, so $(\cup O_\lambda)^c$ is ꝑ-Ƅ closed yielding $\cap O_\lambda^c$ is ꝑ-Ƅ closed. Consequently $\cap C_\lambda$ is ꝑ-Ƅ closed.

Remarks 3.8

(a). ꝑ-Ƅ cl(S) ⊇ S.

(b). ꝑ-Ƅ cl(S) is ꝑ-Ƅ closed.

(c). ꝑ-Ƅ cl(S) is the smallest ꝑ-Ƅ closed set containing in S.

Theorem 3.9

Suppose (Ӿ, $\tau_{ꝑ}$) is a PTS and S ⊆ Ӿ. Then S is ꝑ-Ƅ closed if and only if S = ꝑ-Ƅ cl(S).

Proof

Assume that S is ᵽ-Ƅ closed. Since ᵽ-Ƅ cl(S) = $\bigcap_{\lambda \epsilon I}\{C_\lambda : C_\lambda \supseteq S$ where each C_λ is ᵽ-Ƅ closed$\}$, so S is contained in in each member of this collection. It follows that $\bigcap_{\lambda \epsilon I} C_\lambda = S$. Hence ᵽ-Ƅ cl($S$) = S.

Conversely, suppose that S = ᵽ-Ƅ cl(S), then S is ᵽ-closed, since ᵽ-Ƅ cl(S) is a ᵽ-Ƅ closed set.

Definition 3.10

A subset S of a QTS Ӿ is said to be ɋ-Ƅτ closed if ɋ-Ƅ cl(S) ⊆ U, whenever $S \subseteq U$ and U is ɋ- Ƅ open.

Definition 3.11

A subset S of a PTS Ӿ is said to be *ᵽ-Ƅτ closed* if ᵽ-Ƅ cl(S) ⊆ U, whenever $S \subseteq U$ and U is ᵽ-Ƅ open.

Remark.3.12

(i) The complement of ᵽ-Ƅτ closed set is ᵽ-Ƅτ open set.

(ii) The Intersection of all ᵽ-Ƅτ closed sets of Ӿ containing a subset S of Ӿ is called *ᵽ-Ƅτ closure* of S and is denoted by ᵽ-Ƅτ cl(S). Similarly, the *ᵽ-Ƅτ Interior* of S is the union of all ᵽ-Ƅτ open sets contained in S and is denoted by ᵽ-Ƅτ Int(S).

Theorem 3.13

Every ᵽ-closed set in a PTS (Ӿ, τ_p) is ᵽ-Ƅ closed.

Proof

Let S be a ᵽ-closed set in (Ӿ, τ_p), so S^c is ᵽ-open set. Since $S^c \subseteq$ ᵽ-cl(S^c),

ᵽ-Int(S^c) ⊆ ᵽ-Int[ᵽ-cl(S^c)]. Then $S^c \subseteq$ ᵽ-Int[p-cl(S^c)] ∪ ᵽ-cl[p-Int(S^c)], hence S^c is a ᵽ-Ƅ open set. Consequently, S is ᵽ-Ƅ closed.

Theorem 3.14

Every ᵽ-Ƅ closed set in a PTS (Ӿ, $\tau_ᵽ$) is p- bτ closed.

Proof

Let S be a ᵽ-Ƅ closed set so that ᵽ-Int[ᵽ-cl(S)] ∩ ᵽ-cl[ᵽ-Int(S)] ⊆ S.

Let U be a ꝑ-Ƅ open set and $S \subseteq U$. Then ꝑ-Int[ꝑ-cl(S)] ∩ ꝑ-cl[ꝑ-Int(S)] ⊆ U. Since ꝑ-Ƅ cl(S) is the smallest closed set containing S, so

ꝑ-Ƅ cl(S) = S ∪ ꝑ-Int[ꝑ-cl(S)] ∩ ꝑ-cl[ꝑ-Int(S)] ⊆ S ∪ U ⊆ U. Hence S is ꝑ-Ƅτ closed.

Remark 3.15

Theorem 3.13 and Theorem 3.14 together yield the implication

ꝑ-closed ⇒ ꝑ-Ƅ closed ⇒ ꝑ−Ƅτ closed

However, the reverse implication is not always true as is shown below:

Example 3.16

Consider Ꭓ= $\{p, q, r\}$ with p-topologies $\tau_1 = \{\phi, \{p\}, \{q\}, \{p, q\}, Ꭓ\}$, $\tau_2 = \{\phi, \{p\}, Ꭓ\}$, $\tau_3 = \{\phi, \{q\}, Ꭓ\}$, $\tau_4 = \{\phi, \{r\}, Ꭓ\}$, $\tau_5 = \{\phi, \{p, q\}, Ꭓ\}$.

The sets ϕ, Ꭓ, $\{p, r\}, \{q, r\}$ are ꝑ-closed and ϕ, Ꭓ, $\{p, r\}, \{q, r\}, \{p\}, \{q\}, \{r\}$ are ꝑ-Ƅ closed. The sets $\{p\}$, $\{q\}$ are ꝑ-Ƅ closed but not ꝑ-closed sets. The set $\{p, q\}$ is ꝑ-Ƅτ closed set but it is not a ꝑ-Ƅ closed set.

ꝑ-Ƅ Continuity

We aim at to discuss the idea of ꝑ-Ƅ continuity and prove some related results. Throughout this section Ꭓ and Y denote PTSs.

Definition 4.1

Let (Ꭓ, $\tau_{ꝑ}$) and (Ɏ, $\tau'_{ꝑ}$) where $\tau_{ꝑ} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ and $\tau'_{ꝑ} = (\tau'_1, \tau'_2, \tau'_3, \tau'_4, \tau'_5)$ be two PTSs. A mapping h: Ꭓ ⟶ Ɏ is said to be a *penta-b (or ꝑ-Ƅ) continuous map* if for each ꝑ-Ƅ open set S in Ɏ, $h^{-1}(S)$ is ꝑ-Ƅ open in Ꭓ.

Example 4.2

Let Ꭓ = $\{a, b, c\}$ with $\tau_1 = \{\phi. \{a\}, Ꭓ\}, \tau_2 = \{\phi, \{b\}, Ꭓ\}, \tau_3 = \{\phi, \{c\}, Ꭓ\}$, $\tau_4 = \{\phi, \{a, b\}, Ꭓ\}, \tau_5 = \{\phi, \{a, c\}, Ꭓ\}$ and Ɏ = $\{p, q, r\}$ with $\tau'_1 = \{\phi, \{p\}, Ɏ\}$,

$\tau_2^{/} = \{\phi, \{p\}, \{p, r\}, \text{Ɏ}\}$, $\tau_3^{/} = \{\phi, \{p\}, \{p, q\}, \text{Ɏ}\}$, $\tau_4^{/} = \{\phi, \{q\}, \text{Ɏ}\}$, $\tau_5^{/} = \{\phi, \{r\}, \text{Ɏ}\}$ be PTSs.

Define a map h: Ӿ ⟶ Ɏ by *h(a)* = p, *h(b)* = q, *h(c)* = r. We see that ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, Ӿ, are ꝑ-open sets in Ӿ and $\phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}$, Ɏ are ꝑ-open sets in Ɏ. The ꝑ-Ƅ open sets in are Ӿ, ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$ and ꝑ-Ƅ open sets in Ɏ are Ɏ, ϕ, $\{p\}$, $\{q\}$, $\{r\}$, $p, q\}$, $\{p, r\}$. Since for each ꝑ-Ƅ open set S in Ɏ, $h^{-1}(S)$ is ꝑ-Ƅ open in Ӿ, hence h is ꝑ-Ƅ continuous map.

Definition 4.3

A mapping h: Ӿ ⟶ Ɏ is called *ꝑ-Ƅ continuous at a point* $a \epsilon$ Ӿ, if for each ꝑ-Ƅ open set S containing *h(a)* in *Y*, there exist a ꝑ-Ƅ open set R containing a such that $h(R) \subseteq S$.

Theorem 4.4

A mapping h : Ӿ ⟶ Ɏ is ꝑ-Ƅ continuous if and only if h is ꝑ-Ƅ continuous at every point of *Ӿ*.

Proof

Assume that h is a ꝑ-Ƅ continuous mapping and S is a ꝑ-Ƅ open set containing *h(a)* for any a in Ӿ. Then $h^{-1}(S)$ is a ꝑ-Ƅ open set containing a. Next assume that $h^{-1}(S) = W$. Then $h(W) = S$is a ꝑ-Ƅ open set and so there exists a ꝑ-Ƅ open set W containing a. This shows that h is ꝑ-Ƅ continuous at a. Since a was chosen arbitrary in Ӿ, hence h is ꝑ-Ƅ continuous at every point of Ӿ.

Conversely, assume that h is ꝑ-Ƅ continuous at every point of Ӿ. Let S be a ꝑ-Ƅ open set of Ɏ. If $h^{-1}(S) = \phi$, then it is ꝑ-Ƅ open. Consider any a in $h^{-1}(S)$. Since h is ꝑ-Ƅ continuous at a, hence there exists a ꝑ-Ƅ open set W_a containing a and $h(W_a) \subseteq S$. Let $W = \cup \{W_a : a \epsilon h^{-1}(S)\}$. We prove that $W = h^{-1}(S)$. For $a \epsilon h^{-1}(S) \Rightarrow W_a \subseteq W \Rightarrow a \epsilon W$. If $a \in W$, then $a \epsilon W_a$ for some a and so $h(a) \epsilon S$ $i.e.$ $a \epsilon h^{-1}(S)$. Hence $W = h^{-1}(S)$. Each W_a is ꝑ-Ƅ open, so W is ꝑ-Ƅ open. Therefore $h^{-1}(S)$ is ꝑ-Ƅ open in Ӿ. Consequently h is ꝑ-Ƅ continuous.

Theorem 4.5

A mapping h : Ӿ ⟶ Ɏ is ꝑ-Ƅ continuous if and only if $h^{-1}(S)$ is ꝑ-Ƅ closed in Ӿ for any ꝑ-Ƅ closed set S in Ɏ.

Proof
Let h : Ӿ ⟶ Ɏ be a ᵽ-Ƅ continuous map. Let S be a ᵽ-Ƅ closed set in Ɏ. Then S^c is a ᵽ-Ƅ open set in Ɏ and so by ᵽ-Ƅ continuity of h, $h^{-1}(S^c)$ is ᵽ-Ƅ open in Ӿ ⇒ $[h^{-1}(S)]^c$ is ᵽ-Ƅ open in Ӿ ⇒ $h^{-1}(S)$ is ᵽ-Ƅ closed in Ӿ. Hence $h^{-1}(S)$ is ᵽ-Ƅ closed in Ӿ whenever S is ᵽ-Ƅ closed in *Y*.
Conversely, suppose $h^{-1}(S)$ is ᵽ-Ƅ closed in Ӿ whenever S is ᵽ-Ƅ closed in Ɏ. To prove that h : Ӿ ⟶ Ɏ is a ᵽ-Ƅ continuous map, we assume that S is a ᵽ-Ƅ open set in Ɏ. Then S^c is ᵽ-Ƅ closed in Ɏ ⇒ $h^{-1}(S^c)$ is ᵽ-Ƅ closed in Ӿ ⇒ $[h^{-1}(S)]^c$ is ᵽ-Ƅ closed in Ӿ ⇒ $h^{-1}(S)$ is ᵽ-Ƅ open in Ӿ. Hence h is ᵽ-Ƅ continuous.

Theorem 4.6
A mapping h : Ӿ ⟶ Ɏ is ᵽ-Ƅ continuous lf and only if h[ᵽ-Ƅ cl(S)] ⊆ ᵽ-Ƅ cl[$h(S)$] for all S ⊆ Ӿ.

Proof
Suppose h is a ᵽ-Ƅ continuous map. Since ᵽ-Ƅ cl[$h(S)$] is ᵽ-Ƅ closed in Ɏ, so by
Theorem 4.5, ᵽ-Ƅ cl[$h(S)$] is ᵽ-Ƅ closed in Ӿ.
Note that ᵽ-Ƅ cl[h^{-1}(ᵽ − Ƅ cl($g(S)$))] = h^{-1}[ᵽ − Ƅ cl($h(S)$)]. (*)
Also $h(S)$ ⊆ ᵽ − Ƅ cl[$h(S)$] , $S \subseteq h^{-1}[h(S)]$ ⊆ h^{-1}[ᵽ − Ƅ cl($h(S)$)]. Then ᵽ-Ƅ cl(S)⊆ ᵽ − Ƅ cl[$h^{-1}(p - b$ cl($h(S)$))] = h(ᵽ − Ƅ cl[$h(S)$)] by (*) which yield
h[ᵽ-Ƅ cl(S)] ⊆ ᵽ-Ƅ cl[$h(S)$].
Conversely, let h[ᵽ-Ƅ cl(S)] ⊆ ᵽ-Ƅ cl[$h(S)$] for all S ⊆ Ӿ. Let C be a ᵽ-Ƅ closed set in *Y*, so that ᵽ-Ƅ cl(C) = C. Since $h^{-1}(C)$ ⊆ Ӿ, so by hypothesis, h[ᵽ-Ƅ cl(h^{-1} (C))] ⊆ ᵽ − Ƅ cl[$h(h^{-1}$ (C))] = ᵽ − Ƅ cl(C) = C. Therefore ᵽ-Ƅ cl(h^{-1} (C)) ⊆ h^{-1} (C). But $h^{-1}(C)$ ⊆ ᵽ-Ƅ cl(h (C)) always.
Thus ᵽ-Ƅ cl[$h^{-1}(C)$] =$h^{-1}(C)$ and so $h^{-1}(C)$ is ᵽ-Ƅ closed in Ӿ. Hence h is ᵽ-Ƅ continuous.

ᵽ-Ƅ Homeomorphism

Definition 5.1
A mapping h: Ӿ ⟶ Ɏ is said to be *ᵽ-Ƅ open (resp. ᵽ-Ƅ closed) map* if $h(S)$ is ᵽ-Ƅ open (resp. ᵽ-Ƅ closed) in Ɏ for every ᵽ-Ƅ open (resp. ᵽ-Ƅ closed) set S in Ӿ.

Example 5.2
In Example 4.2, *h* is ᵽ-Ƅ open as well as ᵽ-Ƅ closed map.

Proposition 5.3
A mapping h: Ӿ ⟶ Ɏ is ᵽ-Ƅ continuous if and only if h^{-1} : Ɏ ⟶ Ӿ is ᵽ-Ƅ open map.

Proof
Easy to verify.

Definition 5.4
A bijection h: Ӿ ⟶ Ɏ is called a *penta-Ƅ* (or *ᵽ-Ƅ) homeomorphism*, if *h* is ᵽ-Ƅ continuous and its inverse h^{-1} is ᵽ-Ƅ continuous.
Two PTSs Ӿ and Ɏ are termed as *ᵽ-Ƅ homeomorphic*, if there exist a ᵽ-Ƅ homeomorphism h from Ӿ to Ɏ. Symbolically, we write Ӿ $\overset{\text{ᵽ}-\text{Ƅ}}{\cong}$ Ɏ.

Example 5.5
In Example 4.2, the given map *h* is readily seen to be a ᵽ-Ƅ homeomorphism and Ӿ $\overset{\text{ᵽ}-\text{Ƅ}}{\cong}$ Ɏ.

Conclusion

Khan and Khan (2018) presented the idea of PTS and defined new kinds of open and closed sets including ᵽ-open sets and ᵽ-closed sets in PTSs. Certain properties of p-open sets and p-closed sets were considered. In this work we introduced the idea of ᵽ-Ƅ open sets, ᵽ-Ƅ closed sets, ᵽ- bτ open sets, p-bτ closed sets in PTSs and established a relation among these open and closed sets. We also introduced and studied the idea of ᵽ-Ƅ continuous function and ᵽ-Ƅ homeomorphism in PTSs and proved some related results. It seems possible to carry over the other concepts of single topological spaces such as separation axioms, compactness, connectedness etc. to PTSs.

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