A Simple Introduction to the SM's Method

Sanaullah Mastoi[*], Nabi Bux Kalhoro[†], Ali Bakhsh mugheri[‡], Saeed Ahmed Rajput[§], Wan Ainun mior Othman[**], Kumaresan Nallasamy[††]

Abstract:

SM's Method is an iterative, random step size method, which is focused on the approximate solution. This method is also called as numerical or approximate method. The SM's method is abbreviated from the word Sanaullah Mastoi's Method. The method follows the numerical solution of partial differential equations through the finite-difference method. FD method is based on "grids" or "meshes". The SM's method clearly explained the "Step size" or "Mesh size" or "Grid size", with a specific rule which is randomly generated grids. In this method, the Mesh generation process can be followed through Mathematical programing, or the codes called as matrix laboratory or MATLAB. In this computing platform, the MATLAB 'nrand' commands are used for the random step size. The process of mesh generation does not define or found in any specific formula or principle. The idea or method helps the numerical solutions converging, rapid in solutions with the less computational time and error.

Keyword: SM's Method, Finite difference method, Fractional partial differential equations.

Introductions

The SM's method is abbreviated from *Sanaullah Mastoi's Method* (Mastoi et al., 2022). The numerical solution using SM's method of the differential equations are wide-ranging in the study (Ali, Mastoi, Othman, Khater, & Sohail, 2021; Kumaresan., 2020; Mastoi et al., 2022; Mastoi, Kalhoro, et al., 2021; Mastoi, Mugheri, Kalhoro, & Buller, 2020; Mastoi, Mugheri, et al., 2021; Mastoi, Othman, & Nallasamy., 2020a, 2020b; Othman, 2020). The study focuses on the mathematical application, which may cover the scientific includes

[*] Department of Basic Science and Related Studies, Quaid E Awam University of Engineering, Science and Technology (Campus) Larkana, (77150) Sindh, Pakistan, sanaullah.mastoi@quest.edu.pk

[†] Department of Basic Science and Related Studies, Quaid E Awam University of Engineering, Science and Technology (Campus) Larkana, (77150) Sindh, Pakistan, nabibux75k@gmail.com

[‡] Department of Basic Science and Related Studies, Quaid E Awam University of Engineering, Science and Technology (Campus) Larkana, (77150) Sindh, Pakistan, alibakhshmugheri@yahoo.com

[§] Department of Basic Science and Related Studies, Quaid E Awam University of Engineering, Science and Technology (Campus) Larkana, (77150) Sindh, Pakistan, saeed_rajput82@yahoo.com

[**] Institute of Mathematical Science, Faculty of Science, University of Malaya, Kuala Lumpur (50603) Malaysia, wanainun@um.edu.my

[††] Institute of Mathematical Science, Faculty of Science, University of Malaya, Kuala Lumpur (50603) Malaysia, drnk2008@gmail.com

physics in science, biological, engineering and the computational applications. These main factors in the ranges of scientifical, mathematical, and computers related applications (Gu, Wang, Chen, Zhang, & He, 2017; Samaniego et al., 2020; Smitha & Nagaraja, 2020; Sohail et al., 2020) are interrelated and connected with each other or we can say that strongly tied. The solution based on numerical or numerically solved the partial differential equations (PDE's) involve the quality of applied models without contemplating mathematical and computing factors (M. Z. U. Khan et al., 2021; Sohail et al., 2020). The study or practice shows that the results are not useful or as per our expectations (Ali et al., 2021; Ali, Sohail, & Abdullah, 2020; Ali, Sohail, Usman, et al., 2020; Z. Khan et al., 2021; Mahmood, Md Basir, Ali, Mohd Kasihmuddin, & Mansor, 2019; Rashid et al., 2021; Sohail et al., 2020). The numerous techniques help to solve partial differential equations analytically and numerically (Ali, 2019; Ali, Kamal, & Mohyud-Din, 2012; Barman, Seadawy, Akbar, & Baleanu, 2020; Bashan, Yagmurlu, Ucar, & Esen, 2017; Gu et al., 2017; Islam, Akbar, & Azad, 2018). The FD (Finite-difference), FE (Finite-element), CM (Collocation-Method), SM (Spectral-Method), DDM (Domain-Decomposition-Method), ADM (Adomian-Decomposition-Method) and the novel, newly introduced method which is discussed in this study, which is SM's (Sanaullah-Mastoi's) Method, etc., are various techniques (Andreasen & Huge, 2010; Bar-Sinai, Hoyer, Hickey, & Brenner, 2019; "Chapter 5 Preliminary Review of Finite Difference Methods," 1992; Kumaresan., 2020; Natarajan, 2018; Othman, 2020; Shekari, Tayebi, & Heydari, 2019; Smitha & Nagaraja, 2020; Song, Ooi, & Natarajan, 2018; Thomas, 2013) which are used to solve numerically, this is found in these methods that error is found less with high accuracy. The review of an extensive methods is presented. The exactitude will be comprehensive discussed, but the execution of the method will always be emphasized.

The aim is to bring with you from this technique to various techniques. The concept of this newly developed technique is the knowledge between the concept of theory and the numerical experiences or the practice. Frequently, the engineering, physical of scientific problem are difficult to choose or hesitate to use the method or technique, which is quietly impossible to get the results, or difficult to investigate. It is quite odd that method, which is not comprehensively investigated, we have to go for the numerical experimentations. Whereas such investigations are often necessary for linear problems as well. Usually, it is observed that, we don't even know "what to prove analytically?", till that we have series of experiments.

The numerical methods or technique (SM'S Method), which is developed, analyzed, & implemented. The method is usually be demonstrated using the standard model's equation. These will be the basic PDE's as wave equation, Euler equation, wave equation, heat equation, Poisson equation. We will generally separate the methods to correspond to the different types of equations, using first parabolic equation method then hyperbolic equations methods and finally elliptic equation method. However, we reserve the right to introduce a different type of an equation at any time.

Partial Differential Equation

The Partial differential equation (PDE) is a mathematical equation. Where the mathematical equation involving two or more than two independent variables, an unknown function that depends on those variables, and the partial derivatives of that unknown function with respect to the independent variable (Samaniego et al., 2020; Strauss, 2007; Thomas, 2013). The highest derivative represents the PDES order. The solution of PDE is divided into particular solution and the general solution. In the respective (particular) solution, the function solves the equation, with the solution going to the identity by substituting into equations. The general solutions contain the particular solution of the concern equation (Chen, Fan, & Wen, 2012). The term "exact" or exact solution is generally a imitated solution opposite to approximate solutions. In the solution known specifically for 2nd order & higher order PDEs, we can say that non-linear PDEs. which symbolize the particular solutions. PDEs are the solution of real technical and the scientific problems that follows the PDE definition which includes various variables (Agarwal, Agarwal, & Ruzhansky, 2020; Ahmad, Akgül, Khan, Stanimirović, & Chu, 2020; Duan & Tang, 2020; El-Ajou, Al-Smadi, Oqielat, Momani, & Hadid, 2020; Ghosh, 2020; Habeeb et al., 2020; Harir, Melliani, El Harfi, & Chadli, 2020; Hosseini, Kalhori, & Al-Jumaily, 2020; Kucharski et al., 2020; Miller, 2020; Samaniego et al., 2020; Smitha & Nagaraja, 2020; Sun, Gou, & Geng, 2020; Vargas-González et al., 2020; Wang & Yamamoto, 2020).

Fractional Differential Equations

The fractional differential equation (FODE) is a generalized form of an integer order that can be useful in various areas of defining powers of real number or powers of complex number of the differentiation operator is D.

$$Df(t) = \frac{d}{dt} f(t) \qquad (1)$$

the integral operator J

$$Jf(t) = \int_0^t f(s)\, ds \qquad (2)$$

The development of calculus for such operators that generalized the classical one.

With this in mind, the power describes that the iterative purpose for the linear operator D is transformed into a function f, i.e., often D is written with itself, as in $D^n f(x) = (D \odot \mathrm{D} \odot \mathrm{D} \odot, \dots, \odot \mathrm{D})f$ =(D (D (D (… D (f)…)))). Several researchers focused and concentrated on studying the exact and numerical solutions of the differential equations (Farlow; Strauss) to solve the fractional partial differential equations. The FDE is also known as extraordinary-differential equation, generalized differential equations through fractional calculus.

Numerical Methods

A numerical procedure is stable if all derivatives approximations like (FD- (finite-difference), FE- (finite-elements), FV- (finite-volume), etc.) be likely or close to the exact solution, the grid or step size (like Δx, Δy, Δz, Δt, Δp, Δq etc.) be tends to zero. In addition, the numerical method is stable (like IVPs) if the error is not increasing with the time (or we can say, with iterations).

Iterative Method

The iterative method is an approximation method involving computer mathematics. It is the mathematical process that starting from an initial value, contributes the sequence formation for the improvement solution also known as approximate solutions. The approximate solution to the class of problems. The general or nth approximation is always derived from the previous ones. A specific termination criterion is included in the iterative processing algorithm. An iterative approach is called convergent if the corresponding sequence converges for given initial approximations. Usually, a mathematically rigorous convergence analysis of an iterative approach is performed.

Conversely, to solve the problem, the direct method is used in the finite sequence of operations. The direct method is provided for the exact solution when rounding errors are neglected or ignored, this is like solving the system of linear equations containing $Ax = b$ using the Gaussian elimination method. Iterative methods are only suitable for nonlinear equations. Whereas the iterative methods are useful for linear problems with different variables. The direct methods would be ridiculously difficult (some cases are impossible) even with the best computing power available.

The numerical method is iterative. This method is said to be consistent if approximations such as FDM, FEA. FVM. tends to the size of the exact values. (Chen et al., 2012; Schiesser, 2012).

Finite difference Method

The discussion begins with the numerical methods for the PDEs by getting solved a problem numerically.

$$S_l = AS_{mm}, m \in (a_1, a_2), c > a_1 \tag{3}$$

$$S(d, a_2) = g(d), d \in [a_1, a_2] \tag{4}$$

$$S(a_1, c) = p(c), S(a_2, c) = q(c), c \geq a_1 \tag{5}$$

Here, we have $g(a_1) = p(a_1), and\ g(a_2) = q(a_1)$. We can solve this problem numerically for the mapping of the FDM. The FDM is an approximate technique for the solution of DE or PDE through the derivatives through finite (Step size) differences. The time interval (if apposite) and the spatial domain are discretized into the finite steps. The numerical solution, which the results/solutions/values at the nodes is done by solving algebraic equations. Including finite differences in the step size and his values from neighboring nodes.

FDM converts ODE or PDE, perhaps non-linear, into a system of linear equations and can be solved by matrix algebra techniques. The modern techniques that computers uses can perform these linear algebra calculations efficiently (Grossmann, Roos, & Stynes, 2007). In earlier days, FDM was the most widely uses approach to numerically solve for PDE, including FEM (Dinesh.).

SM's Method

FDM is all about the discretizing geometries into grids or meshes. The SM's method aims to create non-uniform (random) grids. There is no unique method to solve the SM's method as FDM. There is only the process of generating a step size than regular as a normal or equal step size. In this method, the use of mathematical software for the random grids is suggested and recommended. The MATLAB, ANSYS and others used for mesh generation. The name SM's method is abbreviated from Sanaullah Mastoi's Method (Kumaresan., 2020; Mastoi, Kalhoro, et al., 2021; Mastoi, Mugheri, et al., 2020; Mastoi, Mugheri, et al., 2021; Mastoi, Othman, Ali, Rajput, & Fizza, 2021; Mastoi, Othman, et al., 2020a, 2020b; Othman, 2020) .

Material and Methodology

Let us consider the vector equation $F = ma$, where F is the function consider in one dimension can be written in ODE.

$$\text{Second order DE} \qquad m\,\frac{d^2}{dx^2}(x) = F(x) \qquad (6)$$

The Second order ODE can be split by using example, that the mass on a spring in a viscous medium may possibly $F = ku - bv$. Moreover, the velocity and its definition the equation (1) became into (6a & 6b):

$$\frac{d}{dx}(v) = \frac{1}{m} \times F(x) \qquad (6a)$$

Where $\frac{d}{dx}(x) = v$ (6b)

In each equations have general equation or general form as $dy \,/\, dx = f(y, x)$. By using the initial conditions that at $y(x_0) = y_0$, at $x_0 = 0$.

The basic practice of discretization over the range of axes, that is an independent and the dependent variables are shown in the Figure 01.

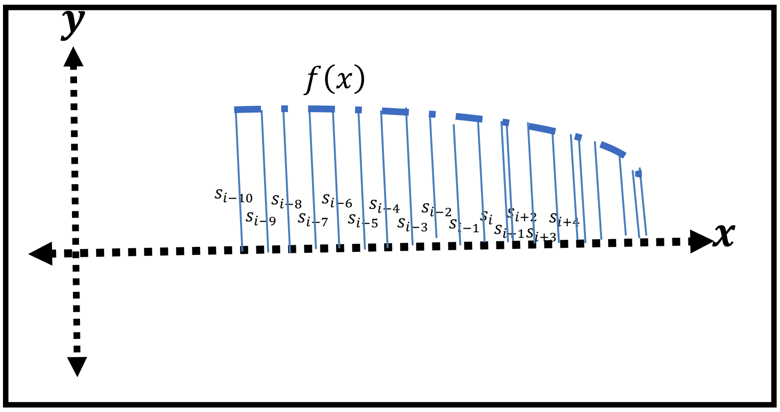

Figure 1. One (1D) dimension discretization

$$\frac{dy}{dx} = f(y, x) \Rightarrow \frac{(y_1 - y_0)}{\Delta x} = f(y_0, x_0) \Longrightarrow y_1 - y_0 = f(y_0, x_0) \times \Delta x$$

$$y_{n+1} = y_n + f(y_n, x_n) \times \Delta x \text{ where } n \geq 0 \qquad (7)$$

The finite discretization for DEs is presented in one dimension, two dimension and three dimensions. The Mesh generation process for the SM's method also describe in one, two and three dimensions. The discretization of the model problems in one dimension, two dimension and three dimensions, there is no special treatment required for the numerical solutions, the FD scheme applied on the models.

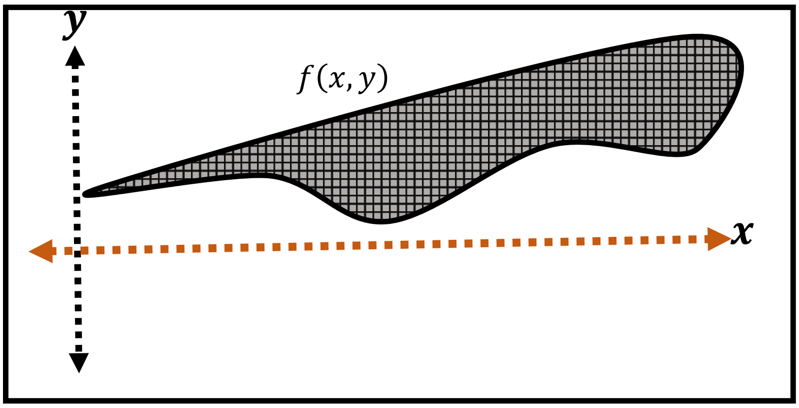

Figure 2. The Two-dimension (2D) discretization

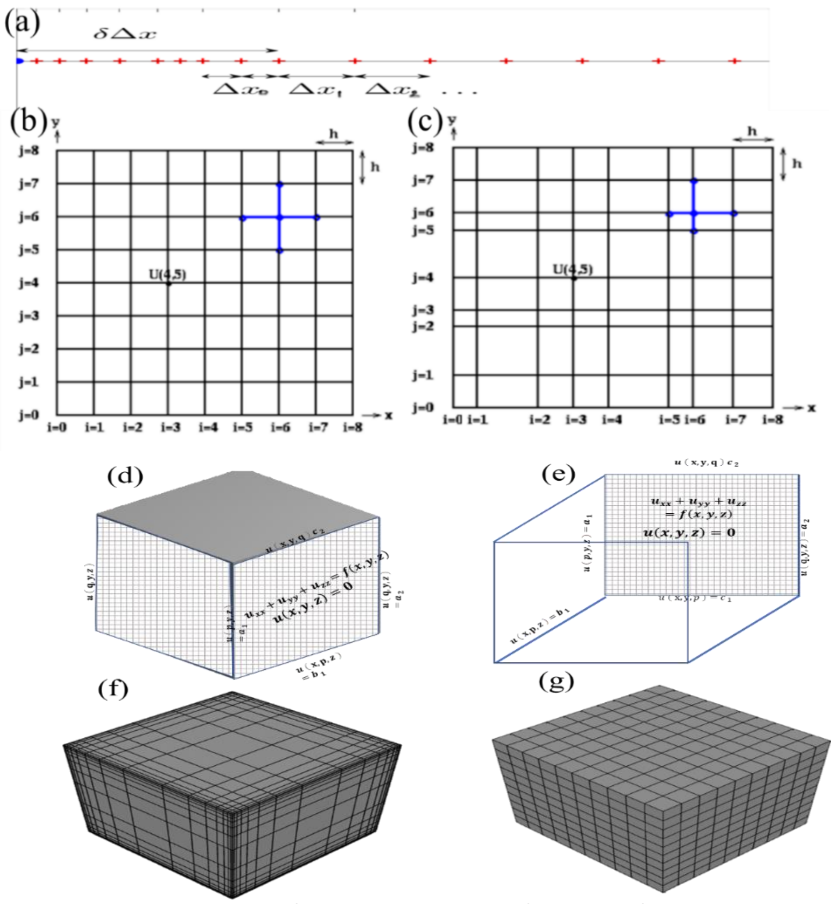

Figure 3. Discretization of 1D, 2D, 3D models specification (Uniform grids, randomly generated grids)

Generation of Uniform Finite Difference Grids and Random grids

The regular spacing grids are generated by writing a code in MATLAB software. The results are Implemented on each interior node by GS iterative method explicitly.

$$u^{p+1}(i,j) = \frac{k^2\times(h^2\times f(i,j)+u^p(i+1,j)-u^p(i-1,j))+h^2\times(u^p(i,j+1)-u^p(i,j-1))}{2(h^2+k^2)}$$

,

Where p and $p+1$ represent iterations and successive iterations, respectively.

The uniform grids are generated with cell size, number of nodes, number of cell size average cell size and standard deviations recorded with MATLAB, using data, and Random grids by varying "nrand" command grids size, we generated the random grids that are shown below.

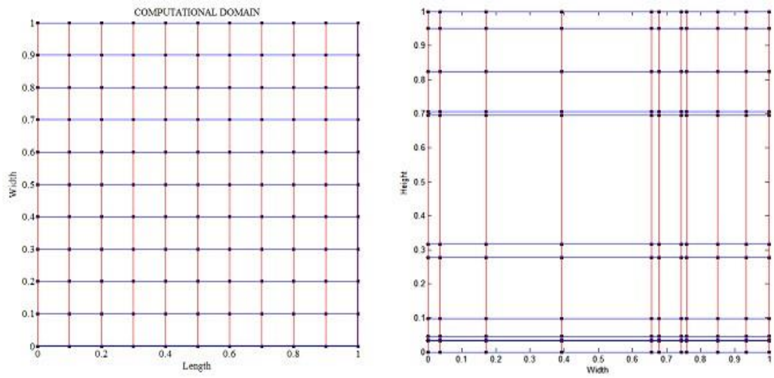

Figure 4. 2D Model Grids (Uniform grids and random grids)

The statistical tests

The regression equation is used for the 8th degree polynomial, where the converging iterations over FDM can be easily predicted by iterations plot using uniform and SM's Method over randomly generated grids. It was clearly mentioned that SM's used fast convergence method with less time and iterations than FDM with uniform mesh generations. The regression equation for an 8th-degree polynomial whose parameters are within the 95% confidence interval. Also, the goodness of fit also shown in Figure 6.

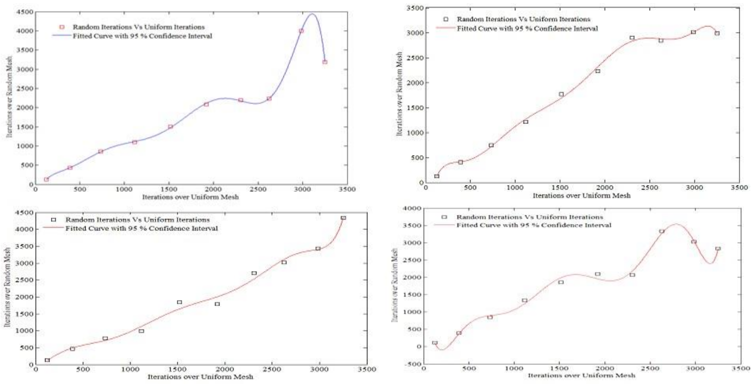

Figure 6. Regression fit for the four realizations of each mesh size

Regression Parameters and Goodness of fit

$$f_1(x) = p_1x^8 + p_2x^7 + p_3x^6 + p_4x^5 + p_5x^4 + p_6x^3 + p_7x^2 + p_8x + p_9$$

Coefficients (with 95% confidence bounds):

p_1 = 9.763e-022 (-6.203e-021, 8.156e-021) , p_2 = -1.273e-017 (-1.095e-016, 8.408e-017)

p_3 = 6.779e-014 (-4.7e-013, 6.056e-013) , p_4 = -1.901e-010 (-1.779e-009, 1.399e-009)

p_5 = 3.018e-007 (-2.384e-006, 2.988e-006) , p_6 = -0.0002717 (-0.002871, 0.002328)

p_7 = 0.1312 (-1.227, 1.489) , p_8 = -28.22 (-358.1, 301.7)

p_9 = 2051 (-2.339e+004, 2.749e+004)

Goodness of fit:

Sum of Square of Error: 6.96e+004

R-square (Square of Coefficient of Correlation): 0.9937

Root Mean Square Error: 263.8

Table 1
Data for SM's Method

Cell Size	Cell size			Variability		Skewness	correlation	Computational time	Time reduction
	Minimum	Maximum	Average	Standard deviation	Variance				
10×10	1×10^{-6}	3.2	0.1	0.81	0.5661	-0.22	0.57	0.001	19%
20×20	2×10^{-7}	4.45	0.04	0.69	0.4761	-0.10	-0.501	0.002	15%
30×30	2.4×10^{-4}	5.81	0.03	0.82	0.6742	-0.07	-0.117	0.14	0.5%
40×40	4.1×10^{-12}	6.2	0.01	0.92	0.8464	-0.05	-0.25	0.21	17%

The behavior and the graph of SM's Method is varied due to the variability in step size. The interior nodes of $u_{i,j}$ defines

$u_{i-1,j}$ $u_{i,j-1}$ $u_{i+1,j}$ & $u_{i,j+1}$. Where, the solution is expected to compute encircled by values.

Conclusions

In this research, the algorithm of SM's Method discussed in detail. The two-dimensional partial differential equation's solved Dirichlet boundary conditions.

The work can be extended in other directions. Mostly, Engineering models have many applications, like air conditioning and cooling system. Financial engineering, medicine and biology system, gene regulatory network, fluid dynamics and Various scientific models will be discretized using randomly generated grids. The idea also can be extended to the finite volume method.

Novel Method

To the best of the authors' knowledge, the proposed method is Randomly generated grids introduced by Sanaullah Mastoi, the proposed method is also knowns as SM's method that is numerical solution is based on FDM using randomly-generated-grids.

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