New Iterative Methods for the Solution of Nonlinear Equations

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Abstracts

The proposed study develops a new iterative techniques to solve highly nonlinear equations of the form $g(x) = 0$. Two techniques are employed, one of which involves the second derivative and the other free from the second derivative. The Taylor series is used to expand the function by substituting $p = -\frac{g''(x)}{2g'(x) - g''(x)}$ and $g''(x_n) = \frac{g'(x_n + g(x_n)) - g'(x_n)}{g(x_n)}$ to get the two algorithms. Furthermore, several examples of nonlinear equations are considered to get the numerical solutions through the derived algorithms. The proposed work is compared with some other iterative techniques like Newton's Method (NM), Adomian Method (AM), Newton Raphson Method (NRM), Chebyshev's method (CM), and Halley's Method (HM). The obtained results are highly accurate, and the convergence is faster than the techniques mentioned. The results are presented in tabular form and discussed which validates the current work and shows the impact of this article.

Keywords**:** Nonlinear Equation; Taylor Expansion; Adomian Methods; Newton's Methods; Newton Raphson Method; Halley's Methods; Chebyshev's Method.

Introduction

In many fields of science and engineering, the use of iterative approaches to solve nonlinear problems has grown significantly in recent years. Problems are measured, mathematically modeled, and then subsequently solved in advanced disciplines such as engineering and physics to understand and address complex phenomena. It might be challenging to find the exact solution to nonlinear equations in one or more variables. Analytical methods for solving such equations are rare and almost nonexistent. The most popular way for obtaining approximate solutions through an iterative process is numerical techniques. The most popular way for obtaining approximate solutions through an iterative process is numerical techniques. The approximate solution of such problems is commonly obtained by numerical iterative approaches such as Newton's method (Avram Sidi, 2006), since it is not always possible to gain its exact solution by the standard algebraic process. Some

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modifications of Newton's method have been developed (Din, 2008; Li & Chen. 2010; Raza & Ahmed, 2012).

Furthermore, nonlinear equations are frequently found in engineering problems. These include equations employing trigonometric, logarithmic, inverse trigonometric, exponential, hyperbolic, and inverse hyperbolic functions, as well as polynomial equations with a degree greater than or equal to three. Transcendental functions are the broad term for these functions. Such equations are difficult to solve since few of them have exact answers. Nonlinear equations can be approximated using a variety of techniques. A novel and effective iterative technique for resolving nonlinear equations is introduced by Chun (2008), which is the improvement of the famous Newton Raphson Method (NRM). Improvement in the convergence of the NRM is discussed in (Junjua et al., 2015).

Numerous researchers worked on the solution of nonlinear phenomena and have developed and illustrated various iterative methods to address this challenge. For instance (Allame & Azad, 2001), proposed the iterative approaches for the solution of $g(x) = 0$, specifically for iterative methods dealing with non-algebraic functions i.e. transcendental. Mitlif (2013) have derived a three-step iterative method for approximating the root of a nonlinear equation. Darvishi & Barati (2007) have introduced a third-order Newton-type method for approximating solutions to nonlinear equations. Furthermore, Wu & Wu (2000) have developed a family of novel iterative techniques and derive formulae which is free from derivatives. These contributions collectively enhanced the field of iterative methods for solving nonlinear phenomena.

Nazeer et al. (2016) explain the modified Halley's method for solving nonlinear equations. They prove that modified HM has a higher order of convergence than Newton's and Halley's Method (HM). Further, Saeed & Aziz (2008) have studied new two and three-step iterative methods for the solution of nonlinear equations. These contributions have significantly advanced the field of nonlinear equation solving. Therefore there are abundant applications where many researchers have illustrated many new iterative methods (Behl et al., 2018; Chun, 2006; Saheya et al., 2016; Daftardar & Jafari, 2006; Panday et al., 2023; Ali & Pan, 2023), to investigate nonlinear equations.

Numerous iterative approaches have been proposed using derivative of first order and free from second derivative (Eskandari, 2010; Noor and Khan, 2012; Li & Zhang, 2010; Thangkhenpau & Panday, 2023), while Yasmin & Junjua (2012) and Ahmed & Hussain (2012) have suggested same derivative free iterative methods for approximating the zeros of the nonlinear equation based on the central – difference and forward – difference approximations to derivatives. Inspired from these studies, the proposed work tries to derive new iterative techniques to investigate highly nonlinear equations and to show their rapid convergence by comparing them with other iterative methods. Several numerical examples are given to demonstrate the performance of the proposed algorithms by comparing them with some other methods. The rest of the article is organized as follows.

The next section is devoted to an in-depth analysis of the newly developed iterative technique, followed by a detailed derivation of Algorithm I and Algorithm II. The application of these proposed Algorithms is then demonstrated through numerical results, where their performance is compared to other well-known iterative methods. The final section gives a brief conclusion of the proposed study.

New Iterative Techniques

Basic Idea

This study presents the development of novel new iterative methods for calculating or approximating solutions for the nonlinear equations of the form:

$$g(x) = 0 \tag{1}$$

Where $g(x)$ is a real-valued function. While Newton's Method is a widely recognized, straightforward, and quite easy iterative algorithm to solve (1). The efficacy is compromised in Newton's Method when the initial guess is too far from the exact solution or when the derivative is close to zero near the required solution. In this study, to address this limitation, we discuss some new iterative methods with three and higher-order convergence which can be used as an alternative to Newton's method. This new technique aims to enhance the accuracy and efficiency of solving nonlinear equations.

This newly proposed algorithm is derived by applying Taylor's Series expansion and, subsequently, eliminating k by introducing a parameter p. The considered examples and comparative analysis indicate that new formulas have advantages over Newton's method, Stephenson's method (Steffensen, 1933), and other methods. Suppose x_j is the exact solution of Equation (1).

Derivation of Algorithm I

Consider the following auxiliary equation.

$$p^2(x_n - x_0)^2 g^2(x_n) - g^2(x_n) = 0 \tag{2}$$

If $x_n - x_0 = k$ then $x_n = x_0 + k$ and Equation (2) becomes:

$$p^2 k^2 g^2(x_0 + k) - g^2(x_0 + k) = 0 \tag{3}$$

From Taylor series expansion:

$$g(x_n+k)=g(x_n)+kg'(x_n)+\frac{k^2}{2!}g''(x_n)+O(k^2),$$
$$g^2(x_n+k)=g^2(x_n)+k^2g'^2(x_n)+2kg(x_n)g'(x_n)+k^2g(x_n)g'^2(x_n)+O(k^2)$$

Now Equation (3) becomes:

$$p^2k^2\big(g^2(x_n)+k^2g'^2(x_n)+2kg(x_n)g'(x_n)+k^2g(x_n)g'^2(x_n)+O(k^2)\big)-\big(g^2(x_n)+k^2g'^2(x_n)+2kg(x_n)g'(x_n)+k^2g(x_n)g'^2(x_n)+O(k^2)\big)=0,$$
$$p^2k^2g^2(x_n)-g^2(x_n)-k^2g'^2(x_n)-2kg(x_n)g'(x_n)-k^2g(x_n)g'^2(x_n)+O(k^2)=0,$$
$$p^2k^2g^2(x_n)-k^2g'^2(x_n)-2kg(x_n)g'(x_n)-g^2(x_n)=0$$
$$(p^2g^2-g'^2)k^2+2gg'k-g^2=0 \tag{4}$$

Equation (4) is quadratic in k. Solving Equation (4) for h:

$$k=\frac{-2gg'\pm\sqrt{(2gg')^2-4(p^2g^2-g'^2)(-g^2)}}{2(p^2g^2-g'^2)}$$
$$=\frac{-2gg'\pm\sqrt{4g^2g'^2+4p^2g^4-4g^2g'^2}}{2(p^2g^2-g'^2)},\ =\frac{-2gg'\pm 2pg^2}{2(p^2g^2-g'^2)},=\frac{gg'\pm pg^2}{p^2g^2-g'^2},$$
$$=\frac{gg'\pm pg^2}{(pg-g')(pg+g')},$$

Thus, we conclude that:

$$k=\frac{g(g'+pg)}{(pg-g')(pg+g')}\ \ or\ \frac{g(g'-pg)}{(pg-g')(pg+g')},$$

This implies that:

$$k=\frac{g}{pg-g'}\quad or-\frac{g}{pg+g'},$$

Using $k=-\frac{g}{pg+g'}$:

$$x_{n+1}=x_n+k,$$
$$x_{n+1}=x_n-\frac{g}{pg+g'}\,,$$

Now $e_n=x_n-x^*$ which implies that $x_n=e_n+x^*$ and hence $g(x_n)=g(e_n+x^*)$.

Now using Taylor's Expansion, we have:

$$g(x_n)=g(x^*)+e_ng'(x^*)+\frac{e_n^2}{2!}g''(x_n)+\frac{e_n^3}{3!}g'''(x_n)+O(e_n^3) \tag{5}$$

As x^* is the root of Equation (1), so $g(x^*)=0$. Therefore Equation (5) implies:

$$g(x_n)=e_ng'(x^*)+\frac{e_n^2}{2!}g''(x^*)+\frac{e_n^3}{3!}g'''(x^*)+O(e_n^3)$$
$$g'(x_n)=g'(x^*)+e_ng''(x^*)+\frac{e_n^2}{2!}g'''(x^*)+\frac{e_n^3}{3!}g^{(iv)}(x^*)+O(e_n^3)$$

Hence:

$$\frac{g(x_n)}{g'(x_n)} = e_n - \frac{g''(x^*)}{2g'(x^*)}e_n^2 + \left(\frac{g''^2(x^*)}{2g'^2(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right)e_n^3 + \left(-\frac{g^{(iv)}(x^*)}{8g'(x^*)} + \frac{g''(x^*).g'''(x^*)}{12g'^2(x^*)} - \frac{g''^3(x^*)}{2g'^3(x^*)}\right)$$

Now e_{n+1} can be written as:

$$e_{n+1} \quad = x_{n+1} - x^*, = x_n - \frac{g(x_n)}{g'(x_n)+pg(x_n)} - x^*,$$

$$= e_n + x^* - \frac{g(x_n)}{g'(x_n)+pg(x_n)} - x^*\ , \ = e_n - \frac{\frac{g(x_n)}{g'(x_n)}}{1+p\frac{g(x_n)}{g'(x_n)}}$$

$$e_{n+1} = e_n - \frac{e_n - \frac{g''(x^*)}{2g'(x^*)}e_n^2 + \left(\frac{g''^2(x^*)}{2g'^2(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right)e_n^3 + \left(-\frac{g^{(iv)}(x^*)}{8g'(x^*)} + 7\,\frac{g''(x^*).g'''(x^*)}{12g'^2(x^*)} - \frac{g''^3(x^*)}{2g'^3(x^*)}\right)e_n^4}{1+p\left(\begin{array}{c} e_n - \frac{g''(x^*)}{2g'(x^*)}e_n^2 + \left(\frac{g''^2(x^*)}{2g'^2(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right)e_n^3 \\ + \left(-\frac{g^{(iv)}(x^*)}{8g'(x^*)} + 7\,\frac{g''(x^*).g'''(x^*)}{12g'^2(x^*)} - \frac{g''^3(x^*)}{2g'^3(x^*)}\right)e_n^4 \end{array}\right)}$$

$$= e_n\left(1 - \frac{A}{B}\right), = e_n\left(\frac{B-A}{B}\right)$$

Were,

$$A = 1 - \frac{g''(x^*)}{2g'(x^*)}e_n + \left(\frac{g''^2(x^*)}{2g'^2(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right)e_n^2 + \left(-\frac{g^{(iv)}(x^*)}{8g'(x^*)} + \frac{g''(x^*).g'''(x^*)}{12g'^2(x^*)} - \frac{g''^3(x^*)}{2g'^3(x^*)}\right)e_n^3$$

and

$$B = 1 + p\left(\begin{array}{c} e_n - \frac{g''(x^*)}{2g'(x^*)}e_n^2 + \left(\frac{g''^2(x^*)}{2g'^2(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right)e_n^3 \\ + \left(-\frac{g^{(iv)}(x^*)}{8g'(x^*)} + 7\ \frac{g''(x^*).g'''(x^*)}{12g'^2(x^*)} - \frac{g''^3(x^*)}{2g'^3(x^*)}\right)e_n^4 \end{array}\right)$$

By substituting $p = -\frac{g''(x)}{2g'(x)-g''(x)}$, we get:

$$Lim_{n\to\infty}\frac{e_{n+1}}{e_n} = \frac{g''(x^*)}{g''(x^*)-2g'(x^*)}\left(\frac{g''^2(x^*)}{2g'(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right) - \left(\frac{g''^2(x^*)}{2g'(x^*)} - \frac{g'''(x^*)}{3g'(x^*)}\right) \quad (6)$$

Now

$$x_{n+1} \quad = x_n - \frac{g(x_n)}{g'(x_n)+pg(x_n)}$$

$$= x_n - \frac{g(x_n)}{g'(x_n)+\left(-\frac{g''(x_n)}{2g'(x_n)-g''(x_n)}\right)g(x_n)}$$

$$= x_n - \frac{g(x_n)\left(2g'(x_n)-g''(x_n)\right)}{g'(x_n)\left(2g'(x_n)-g''(x_n)\right)-g(x_n)g''(x_n)}$$

This implies that:

$$x_{n+1=}x_n - \frac{g(x_n)\left(2g'(x_n)-g''(x_n)\right)}{2g'^2(x_n)-g'(x_n).g''(x_n)-g(x_n)g''(x_n)} \qquad (7)$$

Derivation of Algorithm II

Equation (7) contains the second derivative of $g(x_n)$. To get a formula that is free from $g''(x_n)$, we put $g''(x_n) = \frac{g'(x_n+g(x_n))-g'(x_n)}{g(x_n)}$ in Equation (7) to get:

$$x_{n+1} = x_n - \frac{g(x_n)\left(2g'(x_n) - \frac{g'(x_n+g(x_n))-g'(x_n)}{g(x_n)}\right)}{2g'^2(x_n) - (g'(x_n)+g(x_n))\frac{g'(x_n+g(x_n))-g'(x_n)}{g(x_n)}}$$

$$x_{n+1} = x_n - \frac{2g^2(x_n)g'(x_n) - g(x_n).g'(x_n+g(x_n)) + g(x_n).g'(x_n)}{2g(x_n).g'^2(x_n) - (g'(x_n)+g(x_n))\left(g'(x_n+g(x_n))-g'(x_n)\right)} \quad (8)$$

Equation (8) is now free from the second derivative.

A few cases are considered to verify that the algorithms derived in Equations (7) and (8) converge.

Analysis of Convergence

In this section, we present the analysis of convergence by giving mathematical proof for the order of convergence of the algorithms.

Definition (1):

Convergent Sequence:

Let $\{x_n\}, n \in N$ by an infinite sequence of real numbers. The sequence is said to be convergent to a number x if for any $\epsilon > 0$ there exists a positive integer n such that:

$$|x_n - x| < \epsilon \ or \ Lim_{n\to\infty} x_n = x \ \ or \ x_n \to x \ as \ n \to \infty.$$

If in addition there exist a constant $c \geq 0$, and integer $n_0 \geq 0$, and $p \geq 0$, such that for all $n \geq n_0$,

$$Lim_{n\to\infty} \frac{|x_n-\alpha|}{|x_n-\alpha|^p} = C \neq 0$$

Then $\{x_n: n \in N\}$ is said to converge to α with order at least p. When $e_n = x_n - \alpha$ is the error in the n^{th} iteration, then $\frac{e_{n+1}}{e_n^p} = C$, where p is the order of convergence, is called the error equation.

Applications on Algorithm I & II

This section evaluates the effectiveness and generalization of the proposed Algorithm I [derived in Equation (7)] and Algorithm – II [Equation (8)] by comparing its results to those of other iterative techniques, such as AM, NM, NR, HM, and CM. To validate the proposed Algorithm I & II, five examples of nonlinear equations are considered to compare the convergence rates of each method with those of Algorithm I & II. The results are presented in tabular form, illustrating the number of iterations required for each method. Mathematica software is used to solve these nonlinear equations with the NSolve command.

Example 1:

Consider a nonlinear function (Nazeer et al., 2016)

$$f(x) = x^2 - e^x - 3x + 2$$

When it comes to solving numerical problems, Table 1 compares our new algorithms I & II, which are based on equations (7 & 8 respectively), against existing approaches. To facilitate comparison, we have both analyzed the statistics and displayed the outcomes. One notable difference between our approach and the others is the number of iterations it requires. In this example, we start the initial guess $x_0 = 2$ and reach the solution 0.2575302854 in the 3^{rd} &4^{th} iteration. This shows that the number of iterations required by Algorithm II same as CM and NR but less than the other iterative methods. This results in a considerably faster convergence, which is important for efficiency.

Table 1: Comparison of the results obtained by various numerical techniques with Algorithm I & II for Problem 1

Numerical Methods	Iterations	x_{n+1}
Algorithm I (Equation 7)	3	0.2575302854
Algorithm – II (Equation 8)	4	0.2575302854
Newton's Methods	6	0.2575302854
Adomian Methods	5	0.2575302854
Chebyshev's method	4	0.2575302854
NRM	4	0.2575302854
Halley's Methods	5	0.2575302854

Example 2:

Let us consider a nonlinear function (Nazeer et al., 2016)

$$f(x) = \sin^2 x - x^2 + 1$$

The initial guess for the problem is $x_0 = -1$.

Table 2: Comparison of the results obtained by various numerical techniques with Algorithm I & II for Problem 2

Numerical Methods	Iterations	x_{n+1}
Algorithm 1 (Equation 7)	4	1.4044916482
Algorithm – II (Equation 8)	4	1.4044916482
Newton's Methods	7	1.4044916482
Adomian Methods	5	1.4044916482
Chebyshev's method	5	1.4044916482
NRM	5	1.4044916482
Halley's Methods	4	1.4044916482

Table 2 displays a comparison of the numerical solution of the derived algorithm I & II based on eq (7 & 8) with alternative iterative approaches to show that the present method requires a lesser number of functional evaluations, as compared to other methods. When we examine their outcomes by taking the initial point $x_0 = -1$and getting the solution 1.4044916482, we can see that the derived algorithms and HM have the same number of iterations but have far fewer iterations than the other algorithms, demonstrating its efficiency in convergence.

Example 3:

Consider a nonlinear function

$$f(x) = cosx - x$$

The initial guess for the function is $x_0 = 1.7$

Starting with the initial point $x_0 = 1.7$, we got the same results by all the iterative methods, which is 0.7390851332. Through a comparative analysis of their outcomes, we can observe that from starting the Initial guess $x_0 = 1.7$ to the solution 0.7390851332 the developed method requires significantly fewer iterations than other algorithms, indicating its high convergence efficiency. This means our derived algorithm is efficient at reaching the solution. The results obtained by different iterative methods and derived algorithms are shown in Table 3.

Table 3: Comparison of the results obtained by various numerical techniques with Algorithm I & II for Problem 3

Numerical Methods	Iterations	x_{n+1}
Algorithm 1 (Equation 7)	3	0.7390851332
Algorithm – II (Equation 8)	3	0.7390851332
Newton's Methods	5	0.7390851332
Adomian Methods	4	0.7390851332
Chebyshev's method	4	0.7390851332
NRM	4	0.7390851332
Halley's Methods	4	0.7390851332

Example 4:

Consider a nonlinear function

$$f(x) = xe^{x^2} - \sin^2 x + 3cosx + 5$$

Initial guess $x_0 = -2$

In Table 4, by comparing the results, we show how quickly different methods solve the same problem. By this comparison, in this example, our method (derived algorithm I Equation 7) needs the same number of iterations to find the solution compared to the other methods but the number of iterations for Algorithm II is less than the other methods. We see that by starting the initial guess $x_0 = -2$, we obtain the solution -

1.20764782713. This means our method is more efficient at reaching the answer.

Table 4: Comparison of the results obtained by various numerical techniques with Algorithm I & II for Problem 4

Numerical Methods	Iterations	x_{n+1}
Algorithm – I (Equation 7)	6	-1.20764782713
Algorithm – II (Equation 8)	5	-1.20764782713
Newton's Methods	9	-1.20764782713
Adomian Methods	6	-1.20764782713
Chebyshev's method	6	-1.20764782713
NRM	6	-1.20764782713
Halley's Methods	6	-1.20764782713

Example 5:

Consider a nonlinear function

$$f(x) = e^{x^2+7x-30} - 1$$

Initial guess $x_0 = 3.5$

Table 5 compares the performance of our novel algorithms I & II, which are based on equations (7 & 8), with previous approaches in the solution of numerical problems. The main difference between our algorithm and the others is seen when taking the $x_0 = 3.5$ as an initial guess and reaching solution 3 how many fewer iterations it needs. This has a significant impact on efficiency since it implies it converges considerably faster. It can be observed that the derived Algorithms take a lesser number of iterations compared to other iterative methods.

Table 5: Comparison of the results obtained by various numerical techniques with Algorithm I & II for Problem 5

Numerical Methods	Iterations	x_{n+1}
Algorithm – I	8	3
Algorithm – II (Equation 8)	6	3
Newton's Methods	13	3
Adomian Methods	7	3
Chebyshev's method	8	3
NRM	8	3
Halley's Methods	8	3

Conclusion

In this research paper, we derived new iterative techniques to investigate a highly nonlinear equation to determine their approximate solution. The computation associated with the counterexamples discussed above was performed by using the Mathematica NSolve command. A comprehensive analysis of the numerical test result has revealed the exceptional reliability and efficiency of our proposed algorithms. It is found to be more efficient than the other well-known methods in terms of a lesser number of iterations and convergence speed. Despite the strong nonlinearity of the examples taken into consideration in this work, the algorithms still indicate rapid convergence. The two developed techniques bring simpler nonlinearity problems closer to their exact solution. Several tables are created for comparison to demonstrate the effectiveness of the established algorithms, and the work is compared with some well-known iterative techniques such as NM, AM, NR, and HM. Also, the study on some nonlinear problems has demonstrated the applicability and validity of the presented family of Algorithms by comparing their results with other iterative techniques.

Furthermore, we can say that the family of algorithms that have been shown performs well overall and has a quick rate of convergence, making it a viable substitute for solving nonlinear equations. Furthermore, the aim is to apply the derived schemes to systems of nonlinear equations. The convergence for the system of nonlinear equations will be focused in the future.

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