Analytical Study of Axisymmetric Squeeze Flow of the Slightly Viscoelastic Fluid Film with Slip Effect

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Abstract

This research study presents an analytical solution to the problem of slow axisymmetric squeeze stream formed by a slightly viscoelastic fluid film between two rounded disks due to slip effect. The study uses suited slip boundary conditions to derive the equations of motion as nonlinear systems of partial differential equation. Analytical solutions of ancient governing equations of action have been constructed by Langlois recursive method up to a third order approximation. The mathematical expressions have been generated for velocity components, pressure distributions, and squeezing forces which based on the slip and slightly viscoelastic factors. The acquired results are depicted graphically on different physical parameters. It is observed that the radial velocity, pressure and squeezing force are rising as the viscoelastic parameters increases. Moreover, it is analyzed that the slip parameter reduces pressure and squeezes force while increasing radial velocity at the upper disk, and when the slightly viscoelastic and slip parameters approach zero, the obtained solutions reflect the classical Newtonian fluid findings.

Keywords: Squeeze Flow; Slip Parameter; Viscoelastic Fluid; Recursive Approach; Nonlinear Partial Differential Equations.

Introduction

The squeeze stream has several applications in different fields of science and engineering, including joint lubrication, compression molding of polymers, journal bearings, and damper processing (Hamrock et al., 2004; Venerus, 2018; Yousfi et al., 2013). Squeezing flow is used to investigate the rheological behavior of viscous and viscoelastic fluids (Coussot, 2014; Engmann et al., 2005; Meeten, 2002). Viscoelastic fluids are categorized as differential, integral, and rate type, and these fluids have wide applications like plastic manufacturing, food processing, and

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lubrication theory, and paint industry. Series of papers are written on the squeezing flow using various analytical techniques.

The slip effect has wide applications in engineering, industry, and biosciences, and many researchers have examined the slip effect on viscous fluids with different geometries (Bhatti et al., 2020; Khokhar et al 2023; Dehraj et al., 2020; Shah et al., 2022; H. Ullah et al., 2021). Researchers have worked on determining the analytical and numerical solutions for the Squeeze flow of Newtonian fluid with the slip effect in the last decay (Li et al., 2022; Qayyum et al., 2015; Siddiqui et al., 2010; Siddiqui et al., 2008; Ullah et al., 2014). Laun et al. (1999) has studied the influence of partial slip effect on the motion of squeeze flow for Newtonian and power-law fluids by employing Lubrication approximation theory. In these studies, authors have computed velocity profile only, and little attention is paid to computing others physical quantities like shear stress, pressure distribution, and squeeze force. Analytical and numerical solutions for the squeeze flow of viscoelastic fluids are also examined to obtain asymptotic solutions (Phan-Thien et al., 1985; Phan-Thien et al., 1987). The effects of connective condition and Cattaneo-Christov theory for third order viscoelastic fluid on squeeze flow between two disk is analyzed using Homotopy method (Hayat et al., 2017; Shafiq et al., 2017). Series of papers have been published that are examining the effect of heat generation, chemical reaction, absorption, and Joule dissipation numerically on the Jefferey and Casson viscoelastic fluids for squeeze flow under the slip condition (Noor et al., 2022; Noor et al., 2021). Recently Memon et al. (2023) have constructed the analytical solutions for viscoelastic fluid squeezed between two disk with and without inertia by using recursive approach.

The study of fluid dynamics has advanced significantly, with recent research focusing on the behavior of complex fluids and flow instabilities. Shaikh et al. (2024) has applied integral transform techniques to analyze unsteady fractionalized Oldroyd B fluid flow, enhancing understanding of these complex systems. Additionally, studies on parametric variations in contra-rotating disc systems and wall film cooling in combustion chambers have contributions in optimizing fluid systems in various engineering applications (Bhutto et al., 2024). The investigation regarding oscillating streams in MHD fluid flow adds another layer to understanding heat transfer in magnetic fields. Numerical analysis of flow rates, porous media, and Reynolds numbers affecting the combining and separating of Newtonian fluid flows (Bhutto et al., 2023).

The literature review shows that slip effect for the squeeze flow of slightly viscoelastic fluid between two disks has not been examined yet. The aim of this study is to examine the analytical solution for the motion of a slightly viscoelastic fluid film in the effect of the Navier-slip condition by using a recursive approach. The governing equations of motion for viscoelastic squeezed flow are represented by a nonlinear system of partial differential equations subject to slip boundary conditions. To compute the velocity profile, shear stress, pressure distribution, and squeeze force via the Langlois recursive approach is applied (Langlois, 1963; Langlois, 1964). Using Mathematica, various physical parameters have been visualized.

Mathematical Model

In this study, the axisymmetric squeeze flow of slightly viscoelastic fluid is considered between two disks of the radius $(R)$with constant viscosity (μ) as mentioned in Figure 1. The distance between the two disks is considered to be $2H$ and both disks are approached to each other with constant velocity (V). The cylindrical coordinates are considered to describe the fluid motion and the slip condition is considered at upper disk $z = H$. The effects of inertia and body forces are neglected. Based upon above assumptions the velocity vector is taken as follows:

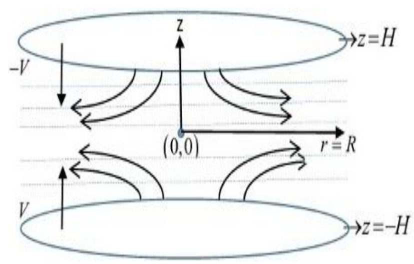


Figure 1: Squeeze flow between two Circular disks (Muravleva, 2018)

$$\vec{V} = [u(r,z), 0, w(r,z)] \tag{1}$$

Based on the following basic governing equations, incompressible, slightly viscoelastic fluids will move in the absence of body force (Kacou et al., 1988; Memon et al., 2022):

$$\nabla . \vec{V} = 0 \tag{2}$$

$$\rho \frac{D\vec{V}}{Dt} = div(\tau) \tag{3}$$

$$\tau = -pI + \mu A_1 + \beta\left(\left| A_1^2 \right|\right) A_1 \tag{4}$$

$$A_1 = \left(grad\,\vec{V}\right) + \left(grad\,\vec{V}\right)^T \tag{5}$$

Here, $\vec{V}$ velocity vector, ρ fluid density, p pressure distribution, τ Cauchy stress tensor, D/Dt material time derivative, μ viscosity, I Identity tensor,

β material constant, $\underset{\sim}{A}_1$ Rivlin-Erickson tensor and $\left|\underset{\sim}{A}_1{}^2\right|$ Trace of the tensor, respectively. The derived mathematical model of axisymmetric squeeze flow of slightly viscoelastic fluid in component form is given in Equations (6-9) obtained by using above assumptions:

$$\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}+\frac{u}{r}=0 \tag{6}$$

$$\frac{\partial p}{\partial r}=\left(\mu+\beta M\right)\left(\nabla^2 u-\frac{u}{r^2}\right)+\beta\left[\frac{\partial M}{\partial z}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)+2\frac{\partial M}{\partial r}\frac{\partial u}{\partial r}\right] \tag{7}$$

$$\frac{\partial p}{\partial z}=\left(\mu+\beta M\right)\left(\nabla^2 w\right)+\beta\left[\frac{\partial M}{\partial r}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)+2\frac{\partial M}{\partial z}\frac{\partial w}{\partial z}\right] \tag{8}$$

Where $M=4\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{u}{r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+2\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)^2$.

The non-zero four stress components are

$$\tau_{rr}=-p+2\left(\mu+\beta M\right)\frac{\partial u}{\partial r};\ \tau_{zr}=\left(\mu+\beta M\right)\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)$$

$$\tau_{\theta\theta}=-p+2\left(\mu+\beta M\right)\frac{u}{r};\ \tau_{zz}=-p+2\left(\mu+\beta M\right)\frac{\partial w}{\partial z} \tag{9}$$

The boundary conditions for the above geometry of the problem are given as follows

$$\tau_{rz}=0,\qquad w=0\quad \text{at}\quad z=0$$

$$w=-\varepsilon V,\qquad u=-\varsigma\frac{\partial u}{\partial z}\quad \text{at}\quad z=H\left(t\right) \tag{10}$$

Where $V(t)=-\frac{dH}{dt}$, ς and ε are the slip coefficient and dimensionless parameter, respectively, but ε varies between $0<\varepsilon\leq 1$. The first and second boundary conditions are taken by imposing the symmetry of the axisymmetric flow at $z=0$. The third and fourth boundary conditions are according to the slip at the upper disk and the upper disk moved with velocity $V(t)$.

Analytical Solution for Flow Variables

The coupled system of nonlinear PDEs given in Equation (6-9) subject to nonhomogeneous boundary conditions, Equation (10) is solved by using the recursive approach proposed by Langlois (Langlois, 1963, 1964). The approximate analytical solution for the velocity components, pressure distribution, and stresses are obtained using Equation (11-14) into Equations (6-10). The linear systems of PDEs (Equations 15-27) of three different approximation orders $O(\varepsilon)$, $O(\varepsilon^2)$and $O(\varepsilon^3)$ gotten by equating terms of equal powers of ε.

The following methods of the flow variables are employed.

$$u(r,z)=\sum_{i=1}^{3}\varepsilon^{(i)}\,u^{(i)}(r,z) \tag{11}$$

$$w(r,z)=\sum_{i=1}^{3}\varepsilon^{(i)}\,w^{(i)}(r,z) \tag{12}$$

$$p(r,z)=\sum_{i=0}^{3}\varepsilon^{(i)}\,p^{(i)}(r,z) \tag{13}$$

$$\underset{\sim}{\tau}(r,z)=\sum_{i=1}^{3}\varepsilon^{(i)}\,\underset{\sim}{\tau}^{(i)}(r,z) \tag{14}$$

Where ε is a small dimensionless number. Replacing the Equations (11-14) in Equations (6-10) and by comparing the coefficients of the same order of ε have found the following three problems.

$O(\varepsilon)$ Problem:

$$\frac{\partial u^{(1)}}{\partial r}+\frac{u^{(1)}}{r}+\frac{\partial w^{(1)}}{\partial z}=0 \tag{15}$$

$$\frac{\partial p^{(1)}}{\partial r}=\mu\left(\nabla^{2}u^{(1)}-\frac{1}{r^{2}}u^{(1)}\right) \tag{16}$$

$$\frac{\partial p^{(1)}}{\partial z}=\mu\,\nabla^{2}w^{(1)} \tag{17}$$

$$\underset{\sim}{\tau}_{rr}^{(1)}=-p^{(1)}+2\mu\frac{\partial u^{(1)}}{\partial r};\;\;\underset{\sim}{\tau}_{rz}^{(1)}=\mu\left(\frac{\partial w^{(1)}}{\partial r}+\frac{\partial u^{(1)}}{\partial z}\right);$$

$$\underset{\sim}{\tau}_{\theta\theta}^{(1)}=-p^{(1)}+2\mu\frac{u^{(1)}}{r};\;\;\underset{\sim}{\tau}_{zz}^{(1)}=-p^{(1)}+2\mu\frac{\partial w^{(1)}}{\partial z} \tag{18}$$

subject to boundary conditions

$$\tau_{rz}^{(1)}=0,\;w^{(1)}=0\;\;\text{at}\;\;z=0;\,w^{(1)}=-\varepsilon V,\;u^{(1)}=-\varsigma\frac{\partial u^{(1)}}{\partial z}\;\;\text{at}\;\;z=H(t) \tag{19}$$

$O(\varepsilon^2)$ Problem:

$$\frac{\partial u^{(2)}}{\partial r}+\frac{u^{(2)}}{r}+\frac{\partial w^{(2)}}{\partial z}=0;\;\frac{\partial p^{(2)}}{\partial r}=\mu\left(\nabla^{2}u^{(2)}-\frac{1}{r^{2}}u^{(2)}\right);\;\frac{\partial p^{(2)}}{\partial z}=\mu\,\nabla^{2}w^{(2)} \tag{20}$$

$$\underset{\sim}{\tau}_{rr}^{(2)}=-p^{(2)}+2\mu\frac{\partial u^{(2)}}{\partial r};\;\;\underset{\sim}{\tau}_{rz}^{(2)}=\mu\left(\frac{\partial w^{(2)}}{\partial r}+\frac{\partial u^{(2)}}{\partial z}\right);$$

$$\underset{\sim}{\tau}_{\theta\theta}^{(2)}=-p^{(2)}+2\mu\frac{u^{(2)}}{r};\;\;\underset{\sim}{\tau}_{zz}^{(2)}=-p^{(2)}+2\mu\frac{\partial w^{(2)}}{\partial z} \tag{21}$$

Subject to conditions

$$\tau_{rz}^{(2)}(0)=0,\ w^{(2)}(0)=0;\text{and } w^{(2)}(H)=-\varepsilon V,\ u^{(2)}(H)=-\varsigma\frac{\partial u^{(2)}}{\partial z} \qquad (22)$$

$O(\varepsilon^3)$ problem:

$$\frac{\partial u^{(3)}}{\partial r}+\frac{u^{(3)}}{r}+\frac{\partial w^{(3)}}{\partial z}=0 \qquad (23)$$

$$\begin{aligned}\frac{\partial p^{(3)}}{\partial r}&=\mu\left(\nabla^2u^{(3)}-\frac{1}{r^2}u^{(3)}\right)+\beta M^{(2)}\left(\nabla^2u^{(1)}-\frac{1}{r^2}u^{(1)}\right)\\&+\beta\left[\frac{\partial M^{(2)}}{\partial z}\left(\frac{\partial w^{(1)}}{\partial r}+\frac{\partial u^{(1)}}{\partial z}\right)+2\frac{\partial M^{(2)}}{\partial r}\frac{\partial u^{(1)}}{\partial r}\right]\end{aligned} \qquad (24)$$

$$\frac{\partial p^{(3)}}{\partial z}=\mu\nabla^2w^{(3)}+\beta\left[M^{(2)}\nabla^2w^{(1)}+\frac{\partial M^{(2)}}{\partial r}\frac{\partial u^{(1)}}{\partial z}+2\frac{\partial M^{(2)}}{\partial z}\frac{\partial w^{(1)}}{\partial z}\right] \qquad (25)$$

$$\begin{aligned}&\tilde{\tau}_{rr}^{(3)}=-p^{(3)}+2\mu\frac{\partial u^{(3)}}{\partial r}+2\beta M^{(2)}\frac{\partial u^{(1)}}{\partial r};\tau_{zr}^{(3)}=\mu\left(\frac{\partial w^{(3)}}{\partial r}+\frac{\partial u^{(3)}}{\partial z}\right)\\&+\beta M^{(2)}\frac{\partial u^{(1)}}{\partial z};\tilde{\tau}_{\theta\theta}^{(3)}=-p^{(3)}+2\mu\frac{u^{(3)}}{r}+\beta M^{(2)}\frac{u^{(1)}}{r};\\&\tilde{\tau}_{zz}^{(3)}=-p^{(3)}+2\mu\frac{\partial w^{(3)}}{\partial z}++\beta M^{(2)}\frac{\partial w^{(1)}}{\partial z}\end{aligned} \qquad (26)$$

Subject to boundary conditions

$$\tau_{rz}^{(3)}(0)=0,\ w^{(3)}(0)=0;\text{ and } w^{(3)}(H)=0,\ u^{(3)}(H)=-\varsigma\frac{\partial u^{(3)}}{\partial z} \qquad (27)$$

Computation of Velocity Components

By transforming the system of PDEs (Equations 15-18) with associated boundary conditions (Equation 19) into a system of stream functions, it has been proposed to model the velocity field of first-order approximation. After analyzing the relationship between the stream function and velocity in Equations (15-19) and eliminating the pressure from the outcome equation, the following Compatibility equation is found, Equation (28) of first-order approximation.

$$E^4\psi^{(1)}(r,z)=0 \qquad (28)$$

Subject to conditions

$$\begin{aligned}&\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial r}=-V,\ \frac{\partial\psi^{(1)}}{\partial z}=-\lambda H\frac{\partial^2\psi^{(1)}}{\partial z^2}\ \text{ at }\ z=H(t);\\&\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial r}\right)=\frac{1}{r}\frac{\partial^2\psi^{(1)}}{\partial z^2},\ \frac{\partial\psi^{(1)}}{\partial r}=0\ \text{ at }\ z=0;\end{aligned} \qquad (29)$$

where $E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, $E^4(*) = E^2(E^2(*))$and $\lambda = \frac{\varsigma}{H}$. The inverse method is used to obtain the solution of the problem (Equations 28-29) by considering the stream function as $\psi^{(1)}(r,z) = r^2 T^{(1)}(z)$ where $T^{(1)}(z)$ is unknown function which will be determined. Thus by using considered stream function it yields:

$$\psi^{(1)}(r,z) = \frac{r^2 V \xi_1}{4}\left(\left(\frac{z}{H}\right)^3 - 3\xi_2\left(\frac{z}{H}\right)\right) \tag{30}$$

$$u^{(1)}(r,z) = \frac{-rV\xi_1}{4H}\left(3\left(\frac{z}{H}\right)^2 - 3\xi_2\right) \tag{31}$$

$$w^{(1)}(r,z) = \frac{V\xi_1}{2}\left(-3\xi_2\left(\frac{z}{H}\right) + \left(\frac{z}{H}\right)^3\right) \tag{32}$$

Where ξ_i's are the slip parameters and defined as $\xi_1 = \frac{1}{1+3\lambda}$; $\xi_2 = 1 + 2\lambda$; $\xi_3 = 1 + 5\lambda$; $\xi_4 = 1 + 7\lambda$; $\xi_5 = 1 + 9\lambda$;. It is noticed that first-order velocity components (Equations 31-32) are matching the result with the creeping squeeze flow of First-grade fluid between two disks with slip conditions (Jasim, 2021).

By reducing the system of PDEs in terms of stream functions, we are able to determine the velocity field of the second-order approximation from Equations (20-21) under homogeneous boundary conditions (Equation 20). The following problem can be derived using the relationship between stream function and velocity.

$$E^4\psi^{(2)}(r,z) = 0 \tag{33}$$

Corresponding to boundary conditions

$$\begin{aligned} &\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial r} = 0,\ \frac{\partial\psi^{(2)}}{\partial z} = -\lambda H\frac{\partial^2\psi^{(2)}}{\partial z^2} \ \text{ at } \ z = H(t); \\ &\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial r}\right) = \frac{1}{r}\frac{\partial^2\psi^{(2)}}{\partial z^2},\ \frac{\partial\psi^{(2)}}{\partial r} = 0 \ \text{ at } \ z = 0; \end{aligned} \tag{34}$$

The solution of Equation (33) becomes zero for any assumptions of stream function due to the homogenous boundary condition and gets the following $\psi^{(2)}(r,z) = 0$,$u^{(2)}(r,z) = 0$ and $w^{(2)}(r,z) = 0$.

The solution of the velocity field of the third-order approximation from Equations (23-26) corresponding to homogeneous boundary conditions (Equation 27) has been computed. Substitution of the first-order solution and the relation of stream function and velocity into Equations (31-32) and then eliminating the pressure by cross differentiating partially obtained the following:

$$\mu\left[\frac{1}{r}\left(E^{4}\psi^{(3)}\right)\right]=\frac{27V^{3}\xi_{1}^{3}r\beta}{H^{6}}\left[12\xi_{2}\left(\frac{z}{H}\right)-19\left(\frac{z}{H}\right)^{3}\right]-\frac{81V^{3}r^{3}\xi_{1}^{3}\beta}{2H^{8}}\left(\frac{z}{H}\right) \tag{35}$$

The following boundary conditions are modified in terms of stream function:

$$\begin{aligned}&\frac{1}{r}\frac{\partial\psi^{(3)}}{\partial r}=0,\ \ \frac{\partial\psi^{(3)}}{\partial z}=-\lambda H\frac{\partial^{2}\psi^{(3)}}{\partial z^{2}}\ \text{ at }\ z=H(t);\\&\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi^{(3)}}{\partial r}\right)=\frac{1}{r}\frac{\partial^{2}\psi^{(3)}}{\partial z^{2}},\ \ \frac{\partial\psi^{(3)}}{\partial r}=0\ \text{ at }\ z=0;\end{aligned} \tag{36}$$

Similarly, the inverse method can be applied for the solution of Equation (35) subject to conditions (Equation 36) by using the following assumptions $\psi^{(3)}(r,z)=\frac{-81V^{3}\xi_{1}^{3}r^{4}}{2H^{8}}\alpha^{(3)}(z)+\frac{27V^{3}\xi_{1}^{3}r^{2}}{H^{6}}\phi^{(3)}(z)$ where $\phi^{(3)}(z)$ and $\alpha^{(3)}(z)$ are unknowns and to be determined. Using the mentioned assumption into Equations (35-36) which leads to ordinary system of differential equations with homogenous boundary conditions, and the solution of system of ordinary differential equations yields the following result:

$$\begin{aligned}\psi^{(3)}(r,z)=&\frac{-27V^{3}r^{4}\beta}{80\mu H^{4}}\left[\xi_{1}^{3}\left(\frac{z}{H}\right)^{5}-2\Lambda_{8}\left(\frac{z}{H}\right)^{3}+\Lambda_{9}\left(\frac{z}{H}\right)\right]\\&+\frac{27V^{3}r^{2}\beta}{1400\mu H^{2}}\left[-25\xi_{1}^{3}\left(\frac{z}{H}\right)^{7}-28\Lambda_{7}\left(\frac{z}{H}\right)^{5}+\Lambda_{10}\left(\frac{z}{H}\right)^{3}-\Lambda_{11}\left(\frac{z}{H}\right)\right]\end{aligned} \tag{37}$$

$$\begin{aligned}u^{(3)}(r,z)=&-\frac{27V^{3}r^{3}\beta}{80\mu H^{5}}\left[-5\xi_{1}^{3}\left(\frac{z}{H}\right)^{5}+6\Lambda_{8}\left(\frac{z}{H}\right)^{2}-\Lambda_{9}\right]\\&+\frac{27V^{3}r\beta}{1400\mu H^{3}}\left[175\xi_{1}^{3}\left(\frac{z}{H}\right)^{6}+140\Lambda_{7}\left(\frac{z}{H}\right)^{4}-3\Lambda_{10}\left(\frac{z}{H}\right)^{2}+\Lambda_{11}\right]\end{aligned} \tag{38}$$

$$\begin{aligned}w^{(3)}(r,z)=&\frac{27V^{3}r^{2}\beta}{20\mu H^{4}}\left[-\xi_{1}^{3}\left(\frac{z}{H}\right)^{5}+2\Lambda_{8}\left(\frac{z}{H}\right)^{3}-\Lambda_{11}\left(\frac{z}{H}\right)\right]\\&+\frac{27V^{3}\beta}{700\mu H^{3}}\left[-25\xi_{1}^{3}\left(\frac{z}{H}\right)^{7}-28\Lambda_{7}\left(\frac{z}{H}\right)^{5}+\Lambda_{10}\left(\frac{z}{H}\right)^{3}-\Lambda_{11}\left(\frac{z}{H}\right)\right]\end{aligned} \tag{39}$$

The third-order approximate solution of the velocity profile and stream function contains terms for slightly viscoelastic parameter β and slip parameters ($\Lambda_{i}, i=1,\cdots,11$)this is the key aspect of the present study and taken as:

$\Lambda_1=\xi_1\xi_3;\Lambda_2=\xi_1\xi_4;\Lambda_3=\xi_1\xi_5;\Lambda_1=\xi_1\xi_3;\Lambda_2=\xi_1\xi_4;\Lambda_4=\xi_1\xi_3-5\xi_2;$
$\Lambda_5=75\Lambda_2+56\Lambda_1\Lambda_4;\Lambda_7=\Lambda_4\xi_1^3;\Lambda_8=\Lambda_1\xi_1^3;\Lambda_6=50\Lambda_3+28\Lambda_2\Lambda_4;$
$\Lambda_{10}=\Lambda_5\xi_1^3;\Lambda_{11}=\Lambda_6\xi_1^3;\Lambda_3=\xi_1\xi_5;\Lambda_9=\Lambda_2\xi_1^3;$

The achieved results by the proposed approach in the absence of a slightly viscoelastic parameter $(\beta=0)$ are strongly agreed with the results of the literature (Jasim, 2021; Li et al., 2022) for squeeze flow of first-grade fluid with slip condition. The solution of the velocity components for $\beta=0$ and slip parameter $(\lambda=0)$ are in excellent coherence with the outcomes of the creeping squeeze flow of viscous fluid between two disks (Lee et al., 1982). If $\lambda=0$, the results of the flow variables are satisfied with (Memon et al., 2022).

Computation of Pressure Distribution

The Equations (40-41) of pressure distribution at first order approximation is obtained from Equations (16-17) by using Equations (31-32).

$$\frac{\partial p^{(1)}}{\partial r}=\frac{-3\mu V\xi_1 r}{2H^3} \tag{40}$$

$$\frac{\partial p^{(1)}}{\partial z}=\frac{3\mu V\xi_1}{H^2}\left(\frac{z}{H}\right) \tag{41}$$

The solution of Equations (40-41) is obtained by directly integration as follows:

$$p^{(1)}(r,z)=\frac{3\mu V\xi_1}{4H}\left[2\left(\frac{z}{H}\right)^2-\frac{r^2}{H^2}\right]+c_1 \tag{42}$$

Where c_1 is the constant of integration.

The second order approximation of pressure distribution is obtained by putting the velocity profile of second order approximation into Equation (20).

$$\frac{\partial p^{(2)}}{\partial r}=0, \qquad \frac{\partial p^{(2)}}{\partial z}=0 \tag{43}$$

After the solution of Equation (43), acquired $p^{(2)}(r,z)=c_2$, where c_2 is the constant of integration. By substituting Equations (38-39) into Equations (24-25) and solving for $p^{(3)}$, it yields the following third order approximation of pressure distribution.

$$p^{(3)}(r,z)=-\frac{81R^4V^3\beta\Lambda_8}{80H^7}+\frac{27V^3\beta R^2}{1400H^5}\left(140\Lambda_{12}\left(\frac{z}{H}\right)^2+\Lambda_{13}\right) \tag{44}$$

$$+\frac{9V^3\beta}{700H^3}\left(3\Lambda_{14}\left(\frac{z}{H}\right)^2-105\Lambda_{15}\left(\frac{z}{H}\right)^4+2030\xi_1^3\left(\frac{z}{H}\right)^6\right)+C^{(3)}$$

The pressure distribution up to third order approximation is obtained by taking the sum of Equation (42-44) and it given as follows:

$$p(r,z)=\frac{27V^3\beta r^2}{1400H^5}\left(140\Lambda_{12}\left(\frac{z}{H}\right)^2+\Lambda_{13}\right)+\frac{3\mu V\xi_1}{4H}\left[2\left(\frac{z}{H}\right)^2-\frac{r^2}{H^2}\right]+C$$
$$-\frac{81r^4V^3\beta\Lambda_8}{80H^7}+\frac{9V^3\beta}{700H^3}\left(3\Lambda_{14}\left(\frac{z}{H}\right)^2-105\Lambda_{15}\left(\frac{z}{H}\right)^4+2030\xi_1^3\left(\frac{z}{H}\right)^6\right) \tag{45}$$

Where $C=C^{(1)}\varepsilon+C^{(2)}\varepsilon^2+C^{(3)}\varepsilon^3$. The constant of integration C is obtained by average boundary condition which is proposed by (Lee et al., 1982)). Thus the complete solution up to third order approximation for pressure distribution is given as follows:

$$p(r,z)=-\frac{81V^3\beta\Lambda_8}{80H^3}\left(\frac{r^4-R^4}{H^4}\right)+\frac{27V^3\beta r^2}{1400H^5}\left(140\Lambda_{12}\left(\frac{z}{H}\right)^2+\Lambda_{13}\right)$$
$$+\frac{9V^3\beta R^2}{2800H^5}\Lambda_{16}+\frac{9V^3\beta}{700H^3}\left(3\Lambda_{14}\left(\frac{z}{H}\right)^2-105\Lambda_{15}\left(\frac{z}{H}\right)^4+2030\xi_1^3\left(\frac{z}{H}\right)^6\right)$$
$$+\frac{9V^3\beta}{2800H^3}\Lambda_{17}+\frac{3\mu V\xi_1}{4H}\left[2\left(\frac{z}{H}\right)^2-\frac{\left(r^2-R^2\right)}{H^2}+2\xi_2-\frac{4}{3}\right]+O\left(\varepsilon^4\right) \tag{46}$$

Where

$$\Lambda_{12}=3\Lambda_8-5\xi_1^3\xi_2;\Lambda_{13}=70\Lambda_9-3\Lambda_{10}-525\xi_1^3\xi_2^2;\Lambda_{15}=4\Lambda_7-2\Lambda_8+75\xi_1^3\xi_2;$$
$$\Lambda_{16}=-1260\Lambda_8+2\left(315\Lambda_9-140\Lambda_{12}-3\Lambda_{13}+105\xi_1^3\right)+700\xi_1^3\xi_2;$$
$$\Lambda_{17}=336\Lambda_7-4\left(3\Lambda_{10}-3\Lambda_{11}+\Lambda_{14}-21\Lambda_{15}+440\xi_1^3\right)+3780\xi_1^3\xi_2-6300\xi_1^3\xi_2^2$$
$$+6300\xi_1^3\xi_2^3;\Lambda_{14}=-70\Lambda_9+3\Lambda_{10}+2625\xi_1^3\xi_2^2;$$

Computation of Normal and Tangential Stresses

The solution for Normal and Tangential stresses correct to third order approximation is found by inserting Equations (31, 32 , 38, 39) and Equation (46) into Equations (18, 21) and Equation (26), respectively, and by adding it yields:

$$\tilde{\tau}_{rr}=\frac{9V^3\beta}{2800H^3}\left(\Lambda_{18}-12\Lambda_{19}\left(\frac{z}{H}\right)^2+140\Lambda_{20}\left(\frac{z}{H}\right)^4-12320\xi_1^3\left(\frac{z}{H}\right)^6\right)$$
$$-\frac{9V^3\beta r^2}{2800H^5}\left(6\Lambda_{21}+420\Lambda_{22}\left(\frac{z}{H}\right)^2-1050\xi_1^3\left(\frac{z}{H}\right)^4\right)+\frac{81V^3\beta\Lambda_8}{80H^7}\left(r^4-R^4\right) \tag{47}$$

$$-\frac{3V\mu\xi_1}{4H}\left[4\left(\frac{z}{H}\right)^2-\left(\frac{r^2-R^2}{H^2}\right)-\frac{4}{3}\right]-\frac{9V^3\beta R^2\Lambda_{16}}{2800H^5}$$

$$\tau_{zr}=-\frac{27V^3\beta r}{700H^5}\left(\Lambda_{23}-70\Lambda_{24}\left(\frac{z}{H}\right)^3+70\xi_1^3\left(\frac{z}{H}\right)^5\right)$$

$$\frac{-3V\mu\xi_1 r}{2H^2}\left(\frac{z}{H}\right)-\frac{81V^3\beta r^3\Lambda_8}{20H^6}\left(\frac{z}{H}\right) \tag{48}$$

Where

$$\Lambda_{18}=12\Lambda_{11}-\Lambda_{17}+6300\xi_1^3\xi_2^3;\Lambda_{19}=3\Lambda_{10}+\Lambda_{14}+1575\xi_1^3\xi_2^2;$$
$$\Lambda_{27}=140\Lambda_9+\Lambda_{13};\Lambda_{20}=12\Lambda_7+3\Lambda_{15}+135\xi_1^3\xi_2;\Lambda_{21}=-105\Lambda_9$$
$$+\Lambda_{13};\Lambda_{22}=9\Lambda_8+2\Lambda_{12}-5\xi_1^3\xi_2;\Lambda_{23}=70\Lambda_9+3\Lambda_{10}+525\xi_1^3\xi_2^2;$$
$$\Lambda_{24}=4\Lambda_7-2\Lambda_8-15\xi_1^3\xi_2;\Lambda_{25}=35\Lambda_9-\Lambda_{13};\Lambda_{26}=3\Lambda_8+2\Lambda_{12}$$
$$-5\xi_1^3\xi_2;;\Lambda_{28}=6\Lambda_8-\Lambda_{12}-5\xi_1^3\xi_2;\Lambda_{32}=\Lambda_{27}-140\Lambda_{28};\Lambda_{29}=24\Lambda_{11}$$
$$+\Lambda_{17}+12600\xi_1^3\xi_2^3;\Lambda_{30}=6\Lambda_{10}-\Lambda_{14}+3150\xi_1^3\xi_2^2;\Lambda_{31}=8\Lambda_7-\Lambda_{15}$$
$$+90\xi_1^3\xi_2;\Lambda_{33}=\Lambda_{29}-12\Lambda_{30}+420\Lambda_{31}-280\xi_1^3;$$

Computation of Normal Squeeze Force

Squeeze flow is characterized by the total force that must be applied on circular disks to compress viscoelastic material between them with constant velocity. The squeeze force at the upper disk is calculated by integrating the negative of the normal axial stress τ_{zz}over the plate surface and it yields:

$$F=\int_0^R -2\pi r\,\tau_{zz}(r,H)\,dr=\frac{3\pi R^4V\xi_1\mu}{8H^3}\left(1+\left(12\xi_2-\frac{20}{3}\right)\left(\frac{H}{R}\right)^2\right)$$

$$+\frac{9\pi R^4V^3\beta}{2800H^5}\left(210\Lambda_8+\left(\Lambda_{16}+3\Lambda_{32}\right)\left(\frac{H}{R}\right)^2+\Lambda_{33}\left(\frac{H}{R}\right)^4\right) \tag{49}$$

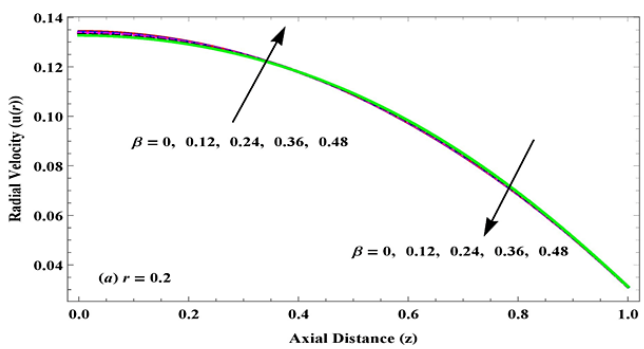
The proposed result is given in Equation (49) for the no-slip case $(\lambda=0)$ and viscoelastic parameter $(\beta=0)$ are satisfying the result of (Lee et al., 1982).

Results and Discussion

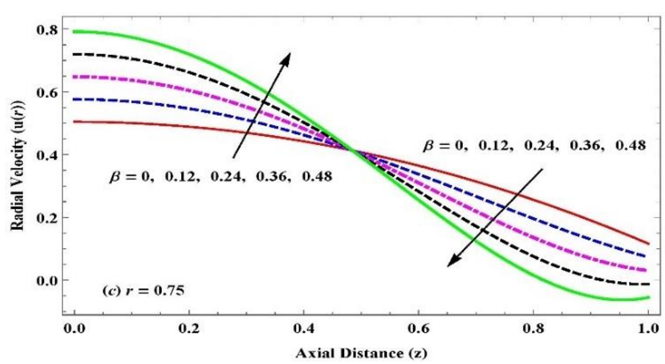
A recursive approach of Langlois is used in this study to obtain an approximate analytical solution for slow squeeze flow of slightly viscoelastic fluid films between two disks with slip effect. Dimensionless variables (50) are introduced to analyze the effects of viscoelastic(β), slip parameter(λ), and radial distance(r) on velocity components, pressure distribution, and normal squeeze forces.

$$\left.\begin{aligned} &r^* = \frac{r}{R};\;\; z^* = \frac{z}{H};\;\; u^* = \frac{H}{VR}u;\;\; w^* = \frac{w}{V};\;\; p^* = \frac{H^3}{\mu V R^2}p; \\ &\delta = \frac{H}{R};\;\; \beta^* = \frac{V^2 R\beta}{\mu H^3};\; \lambda = \frac{\zeta}{H};\;\; F^* = \frac{H^3 F}{\mu R^4 V\pi} \end{aligned}\right\} \quad (50)$$

The behaviors of the flow variables are depicted in Equations (2-8) by the graphical tool of the mathematics-based Mathematica software. A significant effect of the slightly viscoelastic parameter (β) based on the radial velocity (u) including the slip parameter $(\lambda = 0.15)$is shown in Figures 2a-2d along with various radial points $(r = 0.2; 0.5; 0.75$ and $1.0)$. It is stated that the boundary range close to the lower disk is $0 \leq z < 0.5$, while the close to the upper disk is $0.5 \leq z < 1$. It is noted that due to the rise in the slightly viscoelastic parameter the radial velocity increases close to the lower disk $(0 \leq z < 0.5)$, whereas decreases close to the upper disk $(0.5 \leq z < 1)$ and it has the maximum value at the center of the channel $(z = 0)$.



(a)



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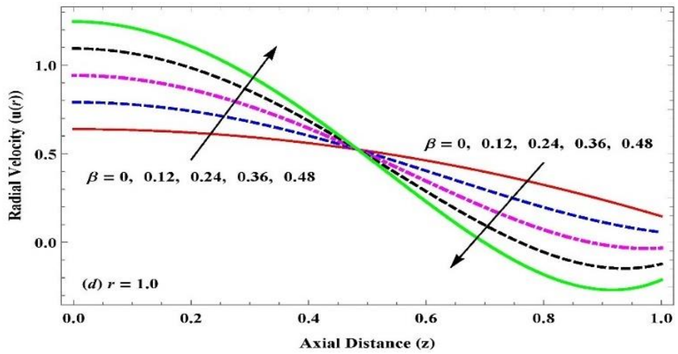

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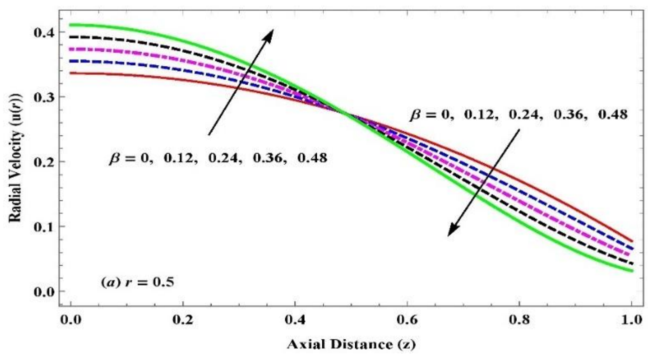
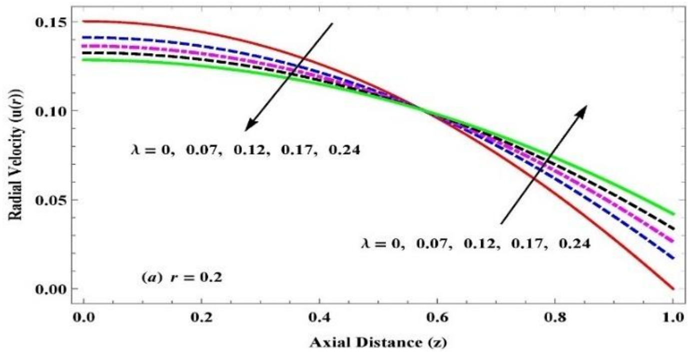


(d)

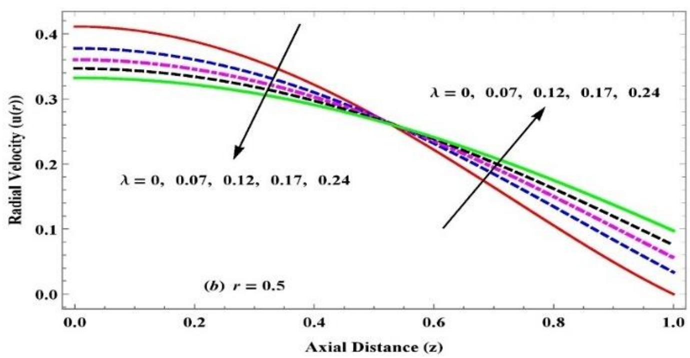
Figure 2: Impact of slightly viscoelastic parameter $(\boldsymbol{\beta})$ ***on Radial velocity with fixed values of*** $\boldsymbol{\delta = 0.1, \lambda = 0.15}$***at the following different radial points (a)*** $\boldsymbol{r = 0.2}$***; (b)*** $\boldsymbol{r = 0.5}$***; (c)*** $\boldsymbol{r = 0.75}$***; (d)*** $\boldsymbol{r = 1.0}$***.***

Physically, the radial velocity of the flow is accelerating when both disks approach the squeezed channel. Conversely, the deceleration is due to the high thickness of material that slows the flow nearer the upper disk. This interprets the behavior of the shear-thickening fluid. Furthermore, the magnitude of the radial velocity at the point $z = 0.5$ has not been impacted due to a rise in slightly viscoelastic parameters and backward flow has occurred on the edges of the channel for $\beta \geq 0.48$. The same behaviour of the radial velocity with variation in β is also demonstrated in the study of Hayat et al. (2017).

Figures 3a-3d illustrate the effect of the slip parameter on the radial velocity of the slightly viscoelastic fluid at different radial points. The influence of λ on radial velocity is surged near the wall of the upper disk and diminished close to the lower disk. Therefore, the increase in λ shows that there is less friction in the vicinity of the upper disk between the surface and the fluid film and in view of that radial velocity escalates in the region $(z < 0.6)$ of the channel. Besides, velocity in the radial direction flows swiftly to the edges.



(a)



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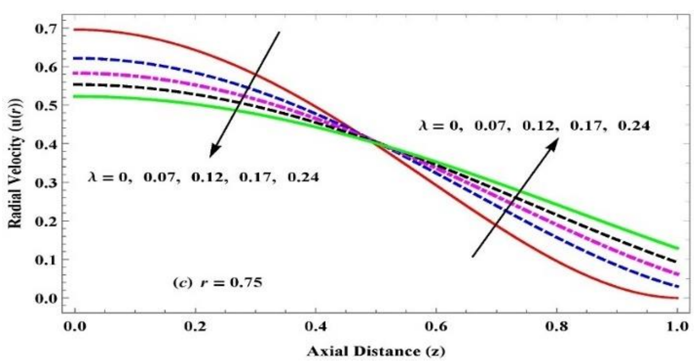
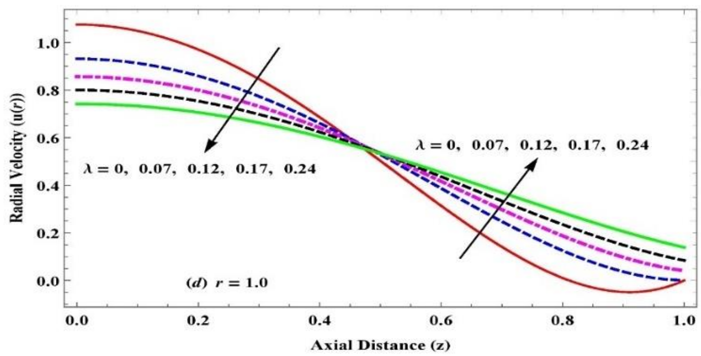


(c)



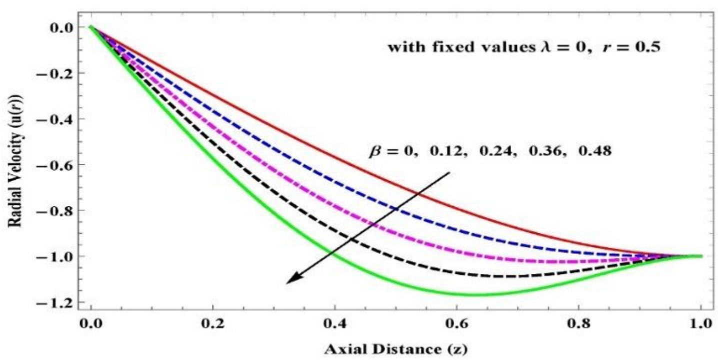
(d)

Figure 3: Impact of slip parameter (λ) on Radial velocity with fixed values of $\delta = \beta = 0.1$ at different radial points (a) $r = 0.2$; (b) $r = 0.5$; (c) $r = 0.75$; (d) $r = 1.0$.

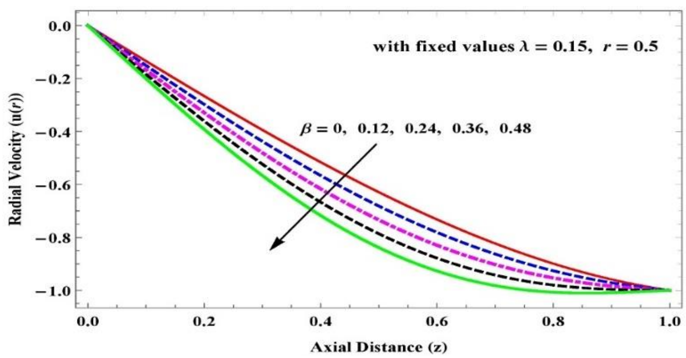
The variation in Axial velocity with slip ($\lambda = 0.15$) and no-slip effects ($\lambda = 0$) due to β is disclosed in Figures 4a and 4b. It is illustrated that in both effects the magnitude of axial velocity has enhanced for β and negativity expressed the flow in the downward direction.

Figures 5a-5b demonstrate the effect of some radial points and slip parameters on axial velocity. It is noticed that the magnitude of axial velocity increases with the rising of radial distance and decreases due to the slip parameter.

Figures 6a and 6b manifest the dimensionless pressure distribution for slightly viscoelastic and viscous fluids on the upper disk to vary the slip parameter (λ). It shows the higher pressure for viscoelastic fluid whereas viscous fluid indicates the opposite, and the slip condition reduces the pressure for both fluids. The radial velocity intensifies along increases β in the vicinity of the upper disk. Therefore, it will decrease the velocity gradient respectively in the region and causes a reduction in pressure.

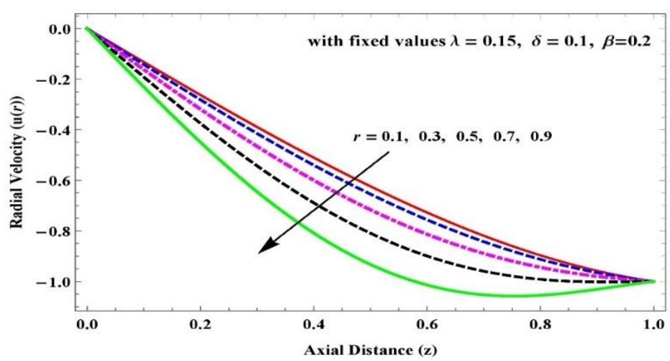


(a)

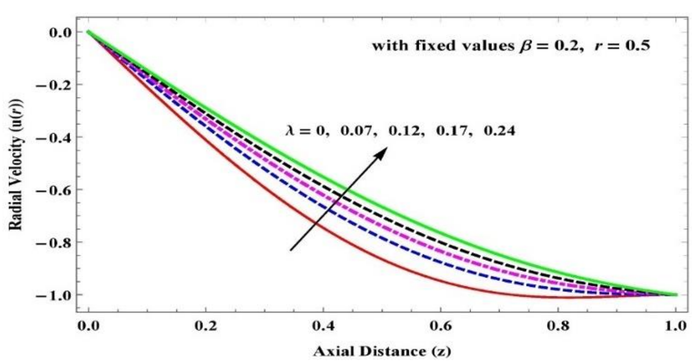


(b)

Figure 4: Impact of slightly viscoelastic parameter (β) on Axial velocity with (a) slip (b) No-slip parameters.

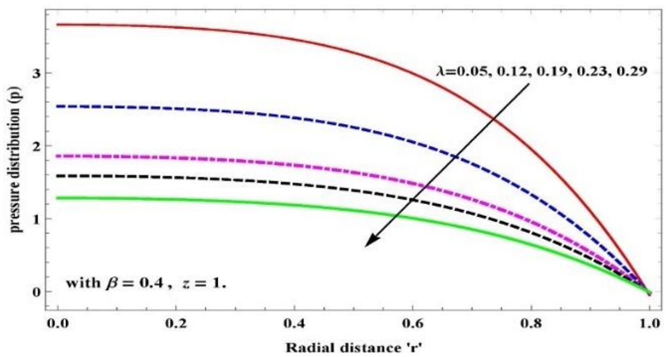


(a)

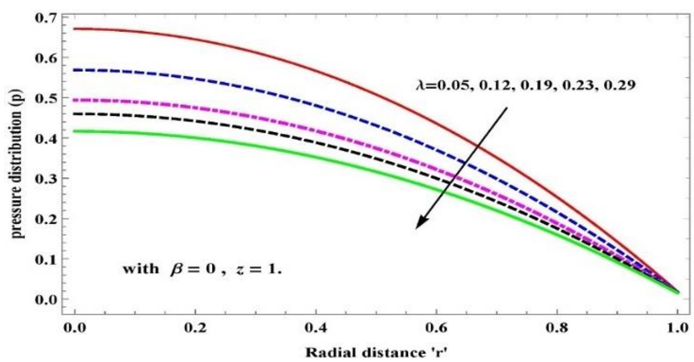


(b)

Figure 5: Impact of (a) radial points (b) slip parameter on Axial velocity for slightly viscoelastic fluid.



(a)

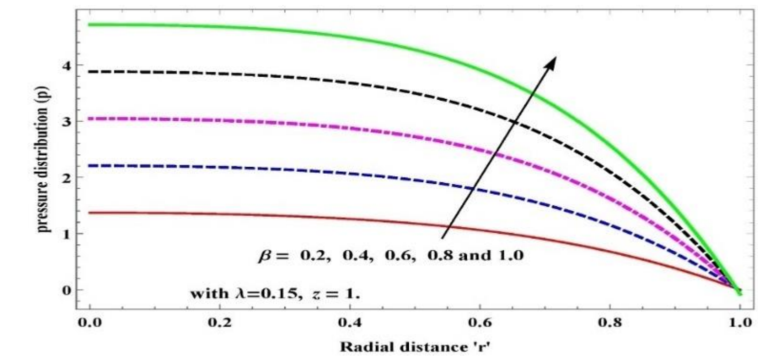


(b)

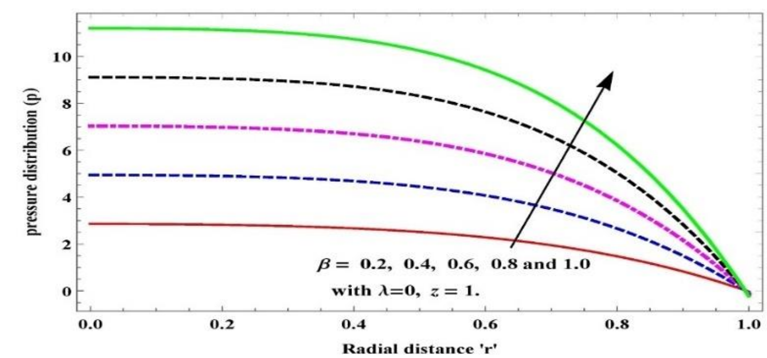
Figure 6: Impact of slip parameter on pressure distribution of (a) Slightly viscoelastic fluid (b) Viscous fluid.

Figures 7a and 7b illustrate the impact of β on the pressure with slip and no-slip conditions. It is shown that higher pressure is required to squeeze the slightly viscoelastic fluid and signified that the fluid is a shear thickening fluid.

The squeeze force is plotted in Figures 8a-8b as the function of the aspect ratio $\left(\delta = \frac{R}{H}\right)$ for examining the influence of the slip parameter and slightly viscoelastic parameter and it declines with the rise of λ but surges due to the rise of β. Indeed, it declines because of the reduction of the friction force between the upper surface and fluid, and it surged due to the thickness of the material. For validation of results, the expressions of flow variables at $(\lambda = 0, \beta = 0)$have identically agreement with the squeeze flow of viscous fluid (Lee et al., 1982).

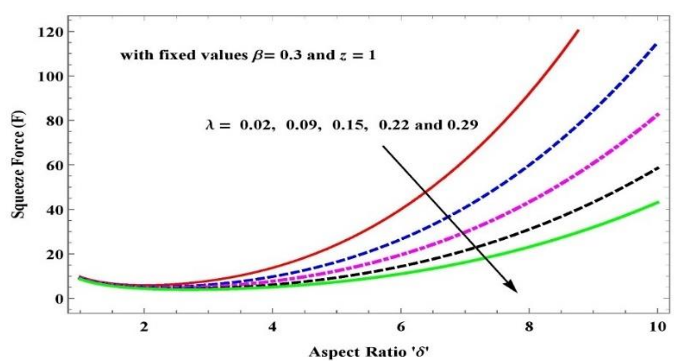


(a)

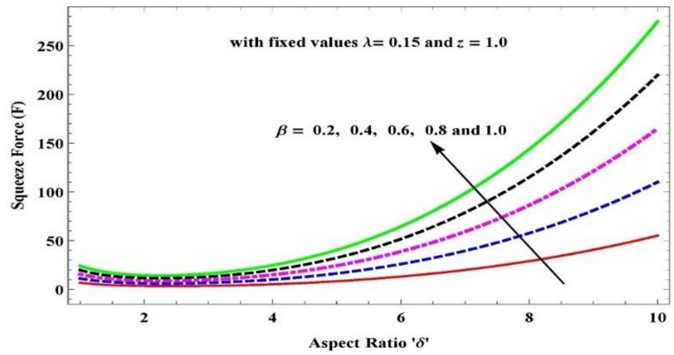


(b)

Figure 7: Impact of slightly viscoelastic parameter on pressure distribution with (a) slip condition (b) No-slip condition.



(a)



(b)

Figure 8: Impact of (a) slip parameter with $\beta = 0.3$ (b) slightly viscoelastic parameter with $\lambda = 0.15$ on squeeze force.

Conclusion

The analytical solution of the proposed model which comprised the non-linear system of PDEs with slip boundary conditions by the Langlois recursive approach has been obtained. It succeeded in getting the analytical expressions up to third-order approximations of the velocity profile, pressure distribution and squeeze force on the upper disk. The impact of pertinent parameters including slip and slightly viscoelastic on radial and axial velocity, pressure distribution and squeeze force have been portrayed graphically. The following appropriate key outcomes are given as follows. The radial velocity diminished for higher values of β in the vicinity of the upper disk, in contrast, it surged close to the lower disk and backward flow occurred on the edges of the channel for $\beta \geq 0.48$. In response to increasing values of the slip parameter(λ), radial velocity accelerated close to the upper disk. When the slightly viscoelastic parameter (β) is extended then the magnitude of axial velocity increases, while the slip parameter (λ) is increased it decreases. More pressure is needed in terms of squeezing the slightly viscoelastic fluid than the Newtonian fluid and the result signified the shear thickening fluid whereas, it drops off as the slip parameter rises. The squeeze force escalates on higher values of β while gradually decreasing as surges in λ.

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