A novel Approach for Real-World Problems Based on the Hermite Interpolation Technique and Analysis Using Basins of Attraction

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Abstract

This paper presents a novel ninth-order iterative scheme based on the Hermite interpolation technique for solving nonlinear equations arises in Real-World models of the form $f(x) = 0$*. In contrast to traditional methods, this approach does not use second derivatives, instead relying on three function evaluations and two first derivative evaluations per iteration. Existing iterative methods frequently suffer from slow convergence and the requirement for higher-order derivatives, which can be computationally costly. The proposed method addresses these limitations by providing a faster convergence without the computational burden of second derivatives. The Taylor series expansion is used to conduct a detailed convergence analysis of the proposed method. The method's effectiveness and stability are further validated by comparisons with existing approaches in the literature.*

Keywords: Hermite Interpolation; Nonlinear Equations; Maple Software; Convergence Analysis; Newton Raphson Method.

Introduction

The real-world application problems in applied sciences are mostly reduced in solving single variable nonlinear equations. These nonlinear equations are solved by Iterative methods instead of direct methods (Ebelechukwu et al., 2018). Various methods have been proposed by researchers in this regard such as (Frontini & Sormani, 2003; Sharma & Bahl, 2017; Soomro et al., 2024). Some researchers proposed bracketing methods (Faraj et al., 2022; Intep, 2018; Jaafar et al., 2019; Kim et al., 2021; Kodnyanko, 2021; Razbani, 2015; Suhadolnik, 2012, 2013). Qureshi et al. (2021) has proposed two step second order method. Li (2019), Li & Jiao (2009), and Soleymani (2011) have proposed fourth order methods. Moreover Khirallah & Alkhomsan (2023) and Zein (2023)

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have proposed fifth order method using weight function without using second derivatives. Thota & Shanmugasundaram (2022) have presented two methods of order six and seven for solution of nonlinear equations. Lakho et al. (2024) have proposed a seventh order method using Lagrange interpolation technique. Sharma & Bahl (2017) have presented an eighth order optimal method for nonlinear equation. Kumar et al. (2013) and Qureshi et al. (2021) have proposed ninth order method. In this study, Hermite interpolation technique is employed to reduce one single derivative in the third stage. The efficiency index (Zein, 2023) ($p^{1/k}$, where p is order of convergence and k is number of functional evaluation) has been boosted. The aim is to propose a new nonlinear method that achieves a better convergence rate while minimizing the number of function evaluations needed per iteration.

Proposed Method

The proposed method using Hermite interpolation technique is derived below using three steps.

For the first step we use Newtons method:

$$q_n = p_n - \frac{f(p_n)}{f'(p_n)} \tag{1}$$

For the second step, we take a variant of Jarratt method.

$$r_n = q_n - \frac{f(q_n)}{f'(p_n)} \frac{5f'^2(p_n)+3f'^2(q_n)}{f'^2(p_n)+7f'^2(q_n)} \tag{2}$$

For the third step, we use the newton method again.

$$p_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)} \tag{3}$$

So, we have a three-step method.

$$\left.\begin{aligned} &\text{Step 1. } q_n = p_n - \frac{f(p_n)}{f'(p_n)} \\ &\text{Step 2. } r_n = q_n - \frac{f(q_n)}{f'(p_n)} \frac{5f'^2(p_n)+3f'^2(q_n)}{f'^2(p_n)+7f'^2(q_n)} \\ &\text{Step 3. } p_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)} \end{aligned}\right\} \tag{4}$$

In (4), there are six function evaluations per iteration, here we approximate $f'(r_n)$ using available data. Since we have four values $f(p)$, $f'(p), f(q)$, $f(r)$ to approximate $f'(r)$ by its Hermite's.

Now let's interpolate polynomial $h_3$of degree 3 at the nodes p, q, r and utilize the approximation $f'(r) \approx h_3'(r)$ in the third step of the iterative scheme. The third degree of Hermite's interpolating polynomial has form

$$h_3(t) = b_0 + b_1(t-p) + b_2(t-p)^2 + b_3(t-p)^3 \tag{5}$$

By taking the derivative of (5) (w.r.t "t").

$$h_3'(t) = b_1 + 2b_2(t-p) + 3b_3(t-p)^2 \tag{6}$$

The conditions' accessible data is used to determine the unknown coefficients.

$$h_3(p) = f(p), \quad h_3(q) = f(q), \quad h_3(r) = f(r) \quad \& \quad h_3'(p) = f'(p)$$

Putting $t = p$ into (5) & (6), we get $b_0 = f(p)$ and $b_1 = f'(p)$. By putting remaining two conditions in (6), $t = q$ & $t = r$, we get a system of two linear equations from which the coefficients b_2 and b_3 are derived.

$$b_2 = \frac{(r-p)f[q,p]}{(r-q)(q-p)} - \frac{(q-p)f[r,p]}{(r-q)(r-p)} - f'(p)\left(\frac{1}{r-p} - \frac{1}{q-p}\right) \tag{7}$$

$$b_3 = \frac{f[r,p]}{(r-q)(r-p)} - \frac{f[q,p]}{(r-q)(q-p)} + \frac{f'(p)}{(r-p)(q-p)} \tag{8}$$

By putting the values of b_1, b_2, b_3 & $t = r$ in (6) we get,

$$h_3'(r_n) = 2(f[p,r] - f[q,p]) + f[q,r] + \frac{q-r}{q-p}\big(f[q,p] - f'(p)\big) \tag{9}$$

Now we replace $f'(r)$ in third step of (11) by (16) $h_3'(r_n)$ and finally we get,

$$\left.\begin{aligned} &\text{Step 1.}\ \ q_n = p_n - \frac{f(p_n)}{f'(p_n)} \\ &\text{Step 2.}\ r_n = q_n - \frac{f(q_n)}{f'(p_n)}\frac{5f'^2(p_n)+3f'^2(q_n)}{f'^2(p_n)+7f'^2(q_n)} \\ &\text{Step 3.}\ \ p_{n+1} = r_n - \frac{f(r_n)}{h_3'(r_n)} \end{aligned}\right\} \tag{10}$$

Equation (10) is our proposed three step scheme.

Convergence Analysis

Theorem

Suppose $\alpha \in D$ represents a simple zero of a function $f\colon D \subset R \to R$, where D is an open interval containing p_0 as an initial approximation of α. In this case, if we consider the method (10), it possesses ninth-order accuracy and requires only five functional evaluations (three functions and two first derivative) per complete iteration.

Proof: Taylor's series expansion of $f(p_n)$.

$$f(p_n) = \sum_{m=0}^{\infty} \frac{f^m(\alpha)}{m!}(p_n - \alpha)^m = f(\alpha) + f'(\alpha)(p_n - \alpha) + \frac{f''(\alpha)}{2!}(p_n - \alpha)^2 + \frac{f'''(\alpha)}{3!}(p_n - \alpha)^3 + \cdots \tag{11}$$

For simplicity, we assume that $A_k = \left(\frac{1}{k!}\right)\frac{f^k(\alpha)}{f'(\alpha)}$, $k \geq 2$.

And assume that $e_n = p_n - \alpha$. Thus, we have

$$f(p_n) = f'(\alpha)\big(e_n + A_2e_n^2 + A_3e_n^3 + A_4e_n^4 + \cdots + O(e_n^{10})\big) \tag{12}$$

$$f'(p_n) = f'(\alpha)\big(1 + 2A_2e_n + 3A_3e_n^2 + 4A_4e_n^3 + \cdots + O(e_n^9)\big) \tag{13}$$

From equation (12) & (13), we get

$$\text{Step 1.}\, q_n = p_n - \frac{f(p_n)}{f'(p_n)} =$$

$$e_n - \frac{f'(\alpha)\left(e_n + A_2e_n^2 + A_3e_n^3 + A_4e_n^4 + \cdots + O(e_n^{10})\right)}{f'(\alpha)\left(1 + 2A_2e_n + 3A_3e_n^2 + 4A_4e_n^3 + \cdots + O(e_n^9)\right)} \quad (14)$$

$$q_n = A_2e_n^2 + (-2A_2^2 + 2A_3)e_n^3 + (4A_2^3 - 7A_2A_3 + 3A_4)e_n^4 + (-8A_2^4 + 20A_2^2A_3 - 6A_3^2 - 10A_2A_4)e_n^5 + \cdots + O(e_n^{10}) \quad (15)$$

$$f(q_n) = f'(\alpha)\begin{pmatrix} A_2e_n^2 + \left(-2{A_2}^2 + 2A_3\right)e_n^3 + \\ (5A_2^3 - 7A_2A_3 + 3A_4)e_n^4 - \\ 2(6A_2^4 - 12A_2^2A_3 + 3A_3^2 + 5A_2A_4)e_n^5 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (16)$$

And

$$f'(q_n) = f'(\alpha)\begin{pmatrix} 1 + 2A_2^2e^2 + 2A_2(-2A_2^2 + 2A_3)e_n^3 + \\ \left(3A_2^2A_3 + 2A_2\begin{pmatrix} 4A_2^3 - \\ 7A_2A_3 + 3A_4 \end{pmatrix}\right)e_n^4 + \\ \begin{pmatrix} 6A_2A_3(-2A_2^2 + 2A_3) - \\ 4A_2\begin{pmatrix} 4A_2^4 - 10A_2^2A_3 + \\ 3A_3^2 + 5A_2A_4 \end{pmatrix} \end{pmatrix}e_n^5 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (17)$$

From (12) and (17), we get

$$\frac{f(q_n)}{f'(p_n)} = A_2e_n^2 + (-4A_2^2 + 2A_3)e_n^3 + (13A_2^3 - 14A_2A_3 + 3A_4)e_n^4 - 2\begin{pmatrix} 19A_2^4 - 32A_2^2A_3 + \\ 6A_3^2 + 10A_2A_4 \end{pmatrix}e_n^5 + \cdots + O(e_n^{10}) \quad 18)$$

From (13) and (17), we get

$$5{f'}^2(p_n) + 3{f'}^2(q_n) =$$

$$f'(\alpha)\begin{pmatrix} 8 + 20A_2e_n + (32A_2^2 + 30A_3)e_n^2 - \\ 4(6A_2^3 - 21A_2A_3 - 10A_4)e_n^3 + \\ \begin{pmatrix} 60A_2^4 - 66A_2^2A_3 + \\ 45A_3^2 + 116A_2A_4 \end{pmatrix}e_n^4 + \\ 24\begin{pmatrix} -6A_2^5 + 9A_2^3A_3 - \\ 5A_2^2A_4 + 5A_3A_4 \end{pmatrix}e_n^5 + \cdots + O(e_n^{10}) \end{pmatrix} \quad (19)$$

And

$${f'}^2(p_n) + 7{f'}^2(q_n) =$$

$$f'(\alpha)\begin{pmatrix} 8+4A_2e_n+(32A_2^2+6A_3)e_n^2- \\ 4(14A_2^3-17A_2A_3-2A_4)e_n^3+ \\ \begin{pmatrix} 140A_2^4-154A_2^2A_3+ \\ 9A_3^2+100A_2A_4 \end{pmatrix}e_n^4+ \\ 8\begin{pmatrix} -42A_2^5+63A_2^3A_3- \\ 35A_2^2A_4+3A_3A_4 \end{pmatrix}e_n^5+\cdots+O(e_n^{10}) \end{pmatrix} \quad (20)$$

From (19) and (20), we get

$$\frac{5f'^2(p_n)+3f'^2(q_n)}{f'^2(p_n)+7f'^2(q_n)}=1+2A_2e_n+ (-A_2^2+3A_3)e_n^2+\frac{1}{2}(-7A_2^3-2A_2A_3+8A_4)e_n^3+ \frac{1}{4}(39A_2^4-67A_2^2A_3+9A_3^2-8A_2A_4)e_n^4+ \frac{1}{8}\begin{pmatrix} -71A_2^5+376A_2^3A_3- \\ 225A_2A_3^2-152A_2^2A_4+48A_3A_4 \end{pmatrix}e_n^5+\cdots+O(e_n^{10}) \quad (21)$$

From (18) and (21), we get

$$\frac{f(q_n)}{f'(p_n)}\frac{5f'^2(p_n)+3f'^2(q_n)}{f'^2(p_n)+7f'^2(q_n)}=A_2e_n^2+(-2A_2^2+2A_3)e_n^3+ (4A_2^3-7A_2A_3+3A_4)e_n^4+\frac{1}{2}\begin{pmatrix} -23A_2^4+42A_2^2A_3- \\ 12A_3^2-20A_2A_4 \end{pmatrix}e_n^5+ \ldots+O(e_n^{10}) \quad (22)$$

From (14) and (22), we get

$$r_n=q_n-\frac{f(q_n)}{f'(p_n)}\frac{5f'^2(p_n)+3f'^2(q_n)}{f'^2(p_n)+7f'^2(q_n)}= \frac{1}{2}(7A_2^4-2A_2^2A_3)e_n^5+\frac{1}{4}\begin{pmatrix} -91A_2^5+107A_2^3A_3- \\ 17A_2A_3^2-4A_2^2A_4 \end{pmatrix}e_n^6+ \ldots+O(e_n^{10}) \quad (23)$$

From (23), we get

$$f(r_n)=f'(\alpha)\begin{pmatrix} \frac{1}{2}(7A_2^4-2A_2^2A_3)e_n^5+ \\ \frac{1}{4}\begin{pmatrix} -91A_2^5+107A_2^3A_3- \\ 17A_2A_3^2-4A_2^2A_4 \end{pmatrix}e_n^6+ \\ \ldots+O(e_n^{10}) \end{pmatrix} \quad (24)$$

From (12) and (24), we get

$$f(p_n)-f(r_n)=$$

$$f'(\alpha)\begin{pmatrix} e_n + A_2e_n^2 + A_3e_n^3 + A_4e_n^4 + \\ \frac{1}{2}(-7A_2^4 + 2A_2^2A_3)e_n^5 + \frac{1}{4}(91A_2^5 - 107A_2^3A_3 + \\ 17A_2A_3^2 + 4A_2^2A_4)e_n^6 + \cdots + O(e_n^{10}) \end{pmatrix} \quad (25)$$

And

$$p_n - r_n = e_n + \frac{1}{2}(-7A_2^4 + 2A_2^2A_3)e_n^5 + \frac{1}{4}\begin{pmatrix} 91A_2^5 - 107A_2^3A_3 + \\ 17A_2A_3^2 + 4A_2^2A_4 \end{pmatrix} e_n^6 + \cdots + O(e_n^{10}) \quad (26)$$

From (25) and (26), we get

$$f[p_n, r_n] = \frac{f(p_n) - f(r_n)}{p_n - r_n} = f'(\alpha)\begin{pmatrix} 1 + A_2e_n + A_3e_n^2 + \\ A_4e_n^3 + \\ \frac{1}{2}(7A_2^5 - 2A_2^3A_3)e_n^5 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (27)$$

From (19) and (23), we get

$$f(q_n) - f(p_n) = f'(\alpha)\begin{pmatrix} -e_n + (-2A_2^2 + A_3)e_n^3 + \\ (5A_2^3 - 7A_2A_3 + 2A_4)e_n^4 - \\ 2\begin{pmatrix} 6A_2^4 - 12A_2^2A_3 + \\ 3A_3^2 + 5A_2A_4 \end{pmatrix} e_n^5 + \\ \begin{pmatrix} 28A_2^5 - 73A_2^3A_3 + \\ 37A_2A_3^2 + 34A_2^2A_4 - \\ 17A_3A_4 \end{pmatrix} e_n^6 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (28)$$

And

$$q_n - p_n = -e_n + A_2e_n^2 + (-2A_2^2 + 2A_3)e_n^3 + (4A_2^3 - 7A_2A_3 + 3A_4){e_n}^4 + \begin{pmatrix} -8A_2^4 + 20A_2^2A_3 - \\ 6A_3^2 - 10A_2A_4 \end{pmatrix} e_n^5 + \begin{pmatrix} 16A_2^5 - 52A_2^3A_3 + 33A_2A_3^2 + \\ 28A_2^2A_4 - 17A_3A_4 \end{pmatrix} e_n^6 + \cdots + O(e_n^{10}) \quad (29)$$

And from (28) and (29)

$$f[q_n, p_n] = \frac{f(q_n) - f(p_n)}{q_n - p_n} =$$

$$f'(\alpha)\begin{pmatrix} 1 + A_2 e_n + (A_2^2 + A_3)e_n^2 + \\ (-2A_2^3 + 3A_2A_3 + A_4)e_n^3 + \\ 2\begin{pmatrix} 2A_2^4 - 4A_2^2A_3 + \\ {A_3}^2 + 2A_2A_4 \end{pmatrix} e_n^4 + \\ \begin{pmatrix} -8A_2^5 + 20A_2^3A_3 - 9A_2A_3^2 - \\ 11A_2^2A_4 + 5A_3A_4 \end{pmatrix} e_n^5 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (30)$$

From (19) and (24), we get

$$(q_n) - f(r_n) = f'(\alpha)\begin{pmatrix} A_2 e_n^2 + (-2A_2^2 + 2A_3)e_n^3 + \\ (5A_2^3 - 7A_2A_3 + 3A_4)e_n^4 + \\ \frac{1}{2}\begin{pmatrix} -31A_2^4 + 50A_2^2A_3 - \\ 12A_3^2 - 20A_2A_4 \end{pmatrix} e_n^5 + \\ \frac{1}{4}\begin{pmatrix} 203A_2^5 - 399A_2^3A_3 + 165A_2A_3^2 \\ +140A_2^2A_4 - 68A_3A_4 \end{pmatrix} e_n^6 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (31)$$

And

$$\begin{gathered} q_n - r_n = A_2 e_n^2 + (-2A_2^2 + 2A_3)e_n^3 + \\ (4A_2^3 - 7A_2A_3 + 3A_4)e_n^4 + \\ \frac{1}{2}\begin{pmatrix} -23A_2^4 + 42A_2^2A_3 - \\ 12A_3^2 - 20A_2A_4 \end{pmatrix} e_n^5 + \\ \frac{1}{4}\begin{pmatrix} 155A_2^5 - 315A_2^3A_3 + 149A_2A_3^2 \\ +116A_2^2A_4 - 68A_3A_4 \end{pmatrix} e_n^6 + \\ \ldots + O(e_n^{10}) \end{gathered} \quad (32)$$

And from (31) and (32)

$$\begin{gathered} f[q_n, r_n] = \frac{f(q_n) - f(r_n)}{q_n - r_n} = \\ f'(\alpha)\begin{pmatrix} 1 + A_2^2 e_n^2 - 2\left(A_2(A_2^2 - A_3)\right) e_n^3 + \\ A_2(4A_2^3 - 6A_2A_3 + 3A_4)e_n^4 + O(e_n^{10}) \end{pmatrix} \end{gathered} \quad (33)$$

And

$$\begin{gathered} \frac{q_n - r_n}{q_n - p_n} = -A_2 e_n + (A_2^2 - 2A_3)e_n^2 + \\ (-A_2^3 + 3A_2A_3 - 3A_4)e_n^3 + \frac{1}{2}\begin{pmatrix} 9A_2^4 - 10A_2^2A_3 + \\ 4A_3^2 + 8A_2A_4 \end{pmatrix} e_n^4 + \\ \frac{1}{4}\begin{pmatrix} -81A_2^5 + 123A_2^3A_3 - 37A_2A_3^2 - \\ 24A_2^2A_4 + 20A_3A_4 \end{pmatrix} e_n^5 + \cdots + O(e_n^{10}) \end{gathered} \quad (34)$$

From (34), (30) & (13), we get

$$\frac{q_n - r_n}{q_n - p_n}\left(\frac{f(q_n) - f(p_n)}{q_n - p_n} - f'(p_n)\right) = f'(\alpha)\begin{pmatrix} A_2^2 e_n^2 - 2\left(A_2(A_2^2 - 2A_3)\right)e_n^3 + \\ 2(2A_2^4 - 5A_2^2A_3 + 2A_3^2 + 3A_2A_4)e_n^4 + \\ \frac{1}{2}\begin{pmatrix} -23A_2^5 + 50A_2^3A_3 - 32A_2A_3^2 - \\ 28A_2^2A_4 + 24A_3A_4 \end{pmatrix} e_n^5 + \\ \frac{1}{4}\begin{pmatrix} 155A_2^6 - 359A_2^4A_3 + 229A_2^2A_3^2 - \\ 32A_3^3 + 136A_2^3A_4 - 176A_2A_3A_4 + 36A_4^2 \end{pmatrix} e_n^6 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (35)$$

From (27), (30), (33) and (35), we get

$$h_3'(r_n) = 2(f[p_n, r_n] - f[q_n, p_n]) + f[q_n, r_n] + \frac{q - r}{q - p}\left(f[q_n, p_n] - f'(p)\right) = f'(\alpha)\begin{pmatrix} 1 + A_2A_4e_n^4 + \begin{pmatrix} 7A_2^5 - 2A_2^3A_3 - \\ 2A_2^2A_4 + 2A_3A_4 \end{pmatrix} e_n^5 + \\ \frac{1}{2}\begin{pmatrix} -91A_2^6 + 107A_2^4A_3 - 17A_2^2A_3^2 + \\ 4A_2^3A_4 - 14A_2A_3A_4 + 6A_4^2 \end{pmatrix} e_n^6 + \\ \ldots + O(e_n^{10}) \end{pmatrix} \quad (36)$$

From (24) and (36), we get

$$\frac{f(r_n)}{h_3'(r_n)} = \frac{f(r_n)}{2(f[p_n, r_n] - f[q_n, p_n]) + f[q_n, r_n] + \frac{q - r}{q - p}\left(f[q_n, p_n] - f'(p)\right)} = \frac{1}{2}(7A_2^4 - 2A_2^2A_3)e_n^5 + \frac{1}{4}\begin{pmatrix} -91A_2^5 + 107A_2^3A_3 - \\ 17A_2A_3^2 - 4{A_2}^2A_4 \end{pmatrix} e_n^6 + \frac{1}{8}\begin{pmatrix} 699A_2^6 - 1468A_2^4A_3 + 629A_2^2A_3^2 - \\ 36A_3^3 + 268A_2^3A_4 - 88A_2A_3A_4 \end{pmatrix} e_n^7 + \cdots + O(e_n^{10}) \quad (37)$$

$$p_{n+1} = r_n - \frac{f(r_n)}{h_3'(r_n)} = \frac{1}{2}A_2(7A_2^4 - 2A_2^2A_3)A_4e_n^9 + \cdots + O(e_n^{10}) \quad (38)$$

Finally, the proposed method (10) has the ninth rate of convergence, requires two first derivatives and three function evaluations each iteration, and has an efficiency index of 1.551845574.

Numerical Experiments and Results

All the problems below are solved using Maple 2022 software on a laptop with the specifications: Intel(R) Core (TM) i3-4010U CPU @ 1.7 GHz and 8.00 GB RAM.

Problem 1: (Van der Waals equation representing a real gas) (Solaiman & Hashim, 2019; Sivakumar & Jayaraman, 2019; Solaiman & Hashim, 2021).
$f_1(p) = 0.986p^3 - 5.181p^2 + 9.067p - 5.289$
Problem 2: The equation that governs the depth of embedment $\boldsymbol{\mu}$ for a sheet-pile wall is expressed as (Naseem et al., 2021; Shams et al., 2022).
$f_2(p) = \frac{p^3+2.87p^2-4.62p-10.28}{4.62}$ s
Problem 3: Fluid permeability problem (Qureshi et al., 2021).
$f_3(p) = 100p^3 - 9.31(1-p)^3$
Newton Raphson Method (NRM) (Akram & Ann, 2012) and the methods proposed by Kumar et al. (2013) and Qureshi et al. (2021) are taken for the comparison.

Table 1. Shows the absolute functional values at 4th iteration of example 1 with initial guess $\boldsymbol{p_0 = 2.03}$.

Method	Solution	Iteration
Proposed Method	6.66287E-3133	4th iteration
NRM	1.24483E-8	
Kumar et al. (2013)	4.89488E-879	
Qureshi et al. (2021)	4.91807E-2491	

Table 2. Shows the number of iterations to reach a fixed error E-3000 of problem 1.

Methods	No of iterations	Error Fixed
Proposed Method	4	E-3000
NRM	13	
Kumar et al. (2013)	5	
Qureshi et al. (2021)	5	

Table 3. Shows the absolute functional values at 4th iteration of example 2 with initial guess $\boldsymbol{p_0 = 3.4}$.

Method	Solution	Iteration
Proposed Method	1.08266E-3019	4th iteration
NRM	1.19720E-5	
Kumar et al. (2013)	5.63861E-834	
Qureshi et al. (2021)	2.48619E-2370	

Table 4. Shows the number of iterations to reach a fixed error E-3000 of problem 2.

Methods	No of iterations	Error Fixed
Proposed Method	4	E-3000
NRM	13	
Kumar et al. (2013)	5	
Qureshi et al. (2021)	5	

Table 5. Shows the absolute functional values at 4th iteration of example 3 with initial guess $p_0 = 0.6$.

Method	Solution	Iteration
Proposed Method	9.70939E-3218	4th iteration
NRM	1.57733E-5	
Kumar et al. (2013)	3.46867E-934	
Qureshi et al. (2021)	3.26183E-2837	

Table 6. Shows the number of iterations to reach a fixed error E-3000 of problem 3.

Methods	No of iterations	Error Fixed
Proposed Method	4	E-3000
NRM	13	
Kumar et al. (2013)	5	
Qureshi et al. (2021)	5	

Various application problems are tested, and the results are shown in Tables 1 to 6, demonstrating that the proposed scheme achieves faster convergence than other counterpart methods and requiring fewer iterations.

Basin of Attraction

This section examines the dynamic behavior of proposed ninth-order iterative method used to solve the nonlinear equation $f(z) = 0$, which involves the function $f: C \rightarrow C$ in a complex plane. Soleymani (2014) is the one who initially thought of and ascribed the basin of attraction for complex Newton's approach.

To visualize the basins of attraction for complex functions, we utilize the efficient computer programming software MATLAB R2014a. Let us consider a rectangle $2D$ various complex plane and consider stopping criteria of threshold 10^{-10} with grid 600×600 points.

Problem 4: $p_1(z) = z^3 - 1,\ p_2(z) = z^4 - 1,\ p_3(z) = z^5 + 1,\ p_4(z) = z^3 - 2z^2 - 9, p_5(z) = z^4 - 4z^3 + 2$

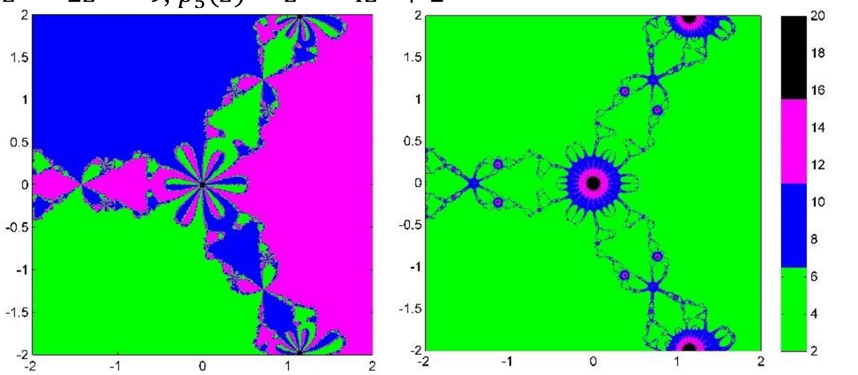

Figure 1. Shows the polynomiographs of $p_1(z)$by proposed method.

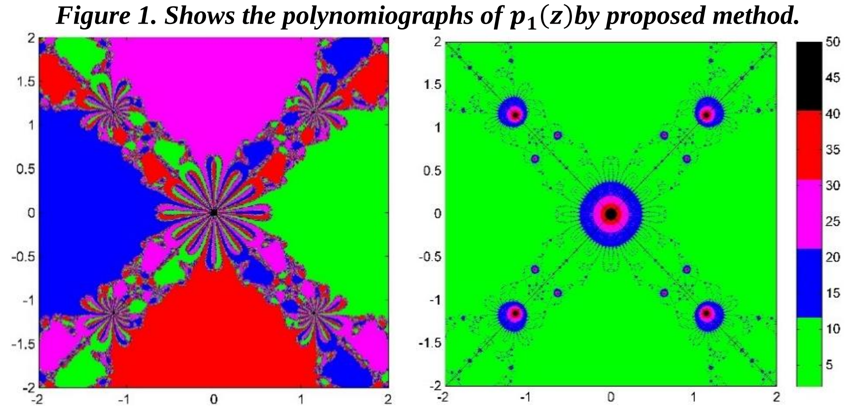

Figure 2. Shows the polynomiographs of $p_2(z)$by proposed method.

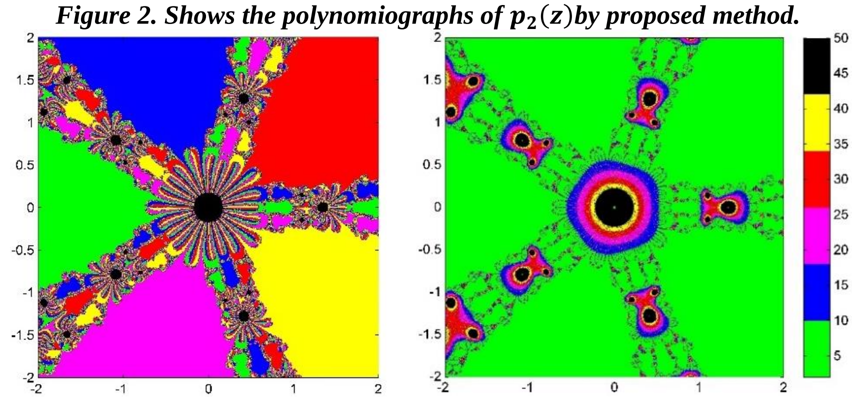

Figure 3. Shows the polynomiographs of $p_3(z)$by proposed method.

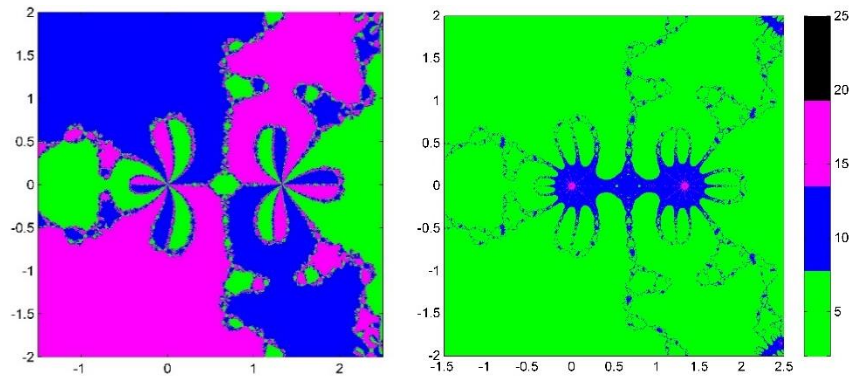

Figure 4. Shows the polynomiographs of $p_4(z)$by proposed method.

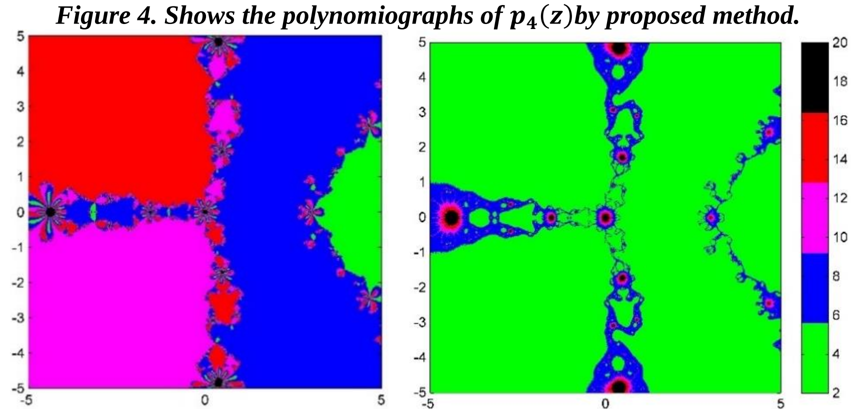

Figure 5. Shows the polynomiographs of $p_5(z)$by proposed method.

To test the stability of the proposed method, we draw the basins of attraction of various functions using the proposed method, as shown in Figures 1 to 5, which clearly demonstrate the stability of the proposed scheme in various problems in complex planes. The left-side Figures shows the roots and right-side figures shows the number of iterations.

Conclusion

In this research, we have developed a ninth-order approach utilizing the Hermite interpolation technique, demonstrating superior convergence performance with fewer iterations. To evaluate the effectiveness of the proposed method against existing techniques in the literature, we applied it to three distinct problems: the van der Waals equation representing a real gas, the depth of embedment (μ) for a sheet-pile wall, and fluid permeability issues. The results show that proposed method consistently outperforms existing methods across all test cases, achieving superior performance with minimal function evaluations.

Additionally, the basin of attraction in the complex plane confirms the stability of the proposed method.

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