Stability Results of Autonomous and Non-singular Delay Difference Equations over Bounded and Unbounded Discrete Intervals

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Abstract

This paper is about the Hyer-Ulam-Rassias stability of a non-singular and non-autonomous difference equation with delay. The details of Hyers-Ulam-rassias is given in the introdution. The main purpose of this paper is to generlaized the previuse results giev in Moonsuwan et al. (2022) from Hyers-Ulam to Hyers-Ulam_Rassias stability. In the first step, the stability results of the mentioned model are achieved over bounded discrete interval. Then in the next step, the corresponding results are achieved over unbounded discrete interval. Furthermore, this concept is extended to infinite impulses. The stability results are achieved with the help of discrete Gronwall inequality. To deal with challenges and achieve desired outcomes, certain assumptions have been introduced.

***Keywords**:* Hyer-Ulam-Rassias; Stabling; Difference equations.

Introduction

Differential equations have significance contribution in both academic and real life. In real life it is used in the field of medical sciences to study the spread of diseases, in the prediction movement of electricity, various exponential growth and decays, to study the motion of waves and pendulum. The use of differential equation in mathematical model is given in Li and Viglialoro (2021). Despite several applications of differential equations it become difficult to solve a system which depends on discrete time variant, for that purpose we need model of difference equations. Difference equation are applicable in many subjects like statistics, engineering, economics and many other sciences. Problems involving neutron diffusion, transport and time-dependent fluid flows, thermo-nuclear reaction and problem containing many simultaneous partial differential equation's solution were being solved by the utilization of difference equation throughout. Difference equation have a lot of use in

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model like in Diblík et al. (2014). Fractional differential equation is a strong mathematical tool in the study of natural sciences and engineering. There application in modeling is in Etemad et al. (2022).

In the last few years, delay differential equations have paid concentration in different areas of linear as well as nonlinear dynamics and have appeared as a useful tool for determining the difficulties of problems of real-life such as population dynamics, infectious diseases, economics and finance and neuronal networks. A differential equation with delay represents a unique subclass within the broader category of functional differential equations; its evolution depends on both the historical states and current state. Today, the analysis of linear as well as nonlinear dynamics of differential equations with delay still have numerous novel problems to scholars. When individuals apply delay differential equations for addressing practical issues, it is the key to be able to fully specify the dynamical properties of differential equations with delay.

Hyers-Ulam stability is the qualitative property of any type of system. Nowadays, it paid the attention of many researchers and they are searching to find the Hyers-Ulam stability of many different type system. This idea was first of all introduced by Ulam (1960) in 1940, when he presented a lecture in the seminar, he put some questions about the group of homomorphism. Hyers (1941), in 1941, answered Ulam question by taking the groups in Banach space and the concept was given the name as Hyers-Ulam stability. Then in 1978, Rassias (1978) generalized this concept known as Hyers-Ulam-Rassias stability. This idea was then extended to differential equation and difference equation by Jung (2010, 2015). Also this idea was used in fractional differential equations by Wang and Xu (2015). The stability in sense of Hyers-Ulam of a nonautonomous and nonsingular difference equation with delay has been examined by Rahmat et al. (2021). In Moonsuwan et al. (2022), the exponential stability, stability in terms of Hyers-Ulam as well as controllability of the nonsingular difference equation with delay has been studied. In Almalki et al. (2022), the stability in sense of β-Hyers-Ulam-Rassias of impulsive difference equations is presented.

We examine the stability in a sense of Hyers-Ulam-Rassias of nonsingular delay difference equation of the form:

$$\begin{cases} \mathbb{C}\mathrm{X}_{n+1} = \mathrm{D}\mathrm{X}_n + \mathrm{E}\mathrm{X}_{n-l} + \mathrm{F}(n, \mathrm{X}_{n-l}), \quad 0.00 \le n, \quad 0.00 \le l, \\ \mathrm{X}_n = \Psi_n, \quad 0.00 \ge n \ge -l \end{cases} \tag{1}$$

Where $\mathbb{C}$, D, and $\mathbb{E}$ are commutable and constant matrices having $n \times n$ order. The matrix C is nonsingular and $\phi \in \mathrm{B}(\mathbb{Z}_+, X)$, a bounded sequences space, also $\mathrm{F} \in \mathcal{C}(\mathrm{Z}_+ \times \mathbb{X}, \mathcal{X})$, a convergent sequences space, where $\mathcal{I} = \{-l, -l + 1.000, \cdots, 0.00\}$.

The stability in Moonsuwan et al. (2022) is given in terms of Hyer-

Ulam, while the proposed study covers more general forms of Hyer-Ulam-Rassias and generalized Hyer-Ulam-Rassias stabilities. The rest of the papers including Rahmat et al. (2021) and Almalki et al. (2022) contains the same stability for some other types of difference equations which shows the credibility and importance of the work in the field of difference equations. In fact, all these results are new in the literature and may open the door for other researchers, working in the field of difference equations and its stability theory.

Organization of the paper is as follows: The 1st section presents the introduction of the proposed work. Preliminaries and basic definitions are covered in the second section of the manuscript. The 3rd section shows the Hyer-Ulam-Rassias stability on a bounded discrete interval, while the 4th section is about the Hyer-Ulam-Rassias stability on an unbounded discrete interval. The 5th section covers the generalized Hyer-Ulam-Rassias stability. The last section gives a brief conclusion of the proposed study.

Preliminaries

For investigating the leading work, we utilize $\mathbb{Z}_+$, $\mathcal{R}^+$ and R, notations for the positive integers, positive real numbers and real numbers and the n-tuples space of $\mathcal{R}$ is represented by R^n. The set $\mathrm{I} = \{0.00, 1.000, \cdots, l\}$ is a proper subset of integers and real numbers. All convergent and sequences that are bounded from $\mathbb{I}$ to X is characterized by $\mathbb{C}(\mathcal{J}, \mathbb{X})$ with

$$\| \mathrm{X} \|_c = \{\sup_{n\in I} \| \mathrm{X}(n) \|, \quad for \quad all \quad \mathrm{X} \in \mathcal{C}(I, X)\}.$$

The following is a non-singular, autonmous and delay differnce systems,

$$\begin{cases} \mathbb{C}\mathrm{X}_{n+1.000} = \mathrm{D}\mathrm{X}_n + \mathrm{E}\mathrm{X}_{n-l} + \mathbb{F}(n, \mathrm{X}_{n-l}), \quad n \geq 0.00, \quad l \geq 0.00, \\ \mathrm{X}_n = \varphi_n, \quad -l \leq n \leq 0.00. \end{cases}$$

Non-singular means that matrix $\mathbb{C}$ is non-singular, autonmous mean all the matrices are free of time n, and delay mean the system has started from some history that is from the time before the currrent time.

Lemma 1: The system

$$\begin{cases} \mathbb{C}\mathrm{X}_{n+1.000} = \mathrm{D}\mathrm{X}_n + \mathrm{E}\mathrm{X}_{n-l} + \mathbb{F}(n, \mathrm{X}_{n-l}), \quad n \geq 0.00, \quad l \geq 0.00, \\ \mathrm{X}_n = \varphi_n, \quad -l \leq n \leq 0.00, \end{cases} \tag{2}$$

has the solution

$$\mathrm{X}_n = D^n C^{-n} \Phi_{0.00} + D^{n-1.000} C^{-n} \sum_{i=0.00}^{l} D^{-i} C^i (E\Phi_{i-l} + \mathbb{F}(i, \Phi_{i-l})) + D^{n-1.000} C^{-n} \sum_{i=l+1.000}^{n} D^{-i} C^i (E\mathrm{X}_{i-l} + \mathrm{F}(i, \mathrm{X}_{i-l})),$$

where $DE = ED$, $EC = CE$ and $DC = CD$.

The proof can be readily acquired through the repeated iteration of 'n'.

Definition 2

Let $\epsilon > 0.00, \Psi > 0.00$ and $\Psi_n \in \mathcal{B}(\mathrm{I}, \mathbb{X})$. G'_n is called an ϵ-approximate solution of (2), if

$$\begin{cases} \| CX'_{n+1.000} - DX'_n - EX'_{n-k} - f(n, X'_{n-k}) \| \leq \epsilon \Psi_n, & n \geq 0.00, \\ \| X'_n - \Phi_n \| \leq \epsilon \Psi, & k \in J. \end{cases}$$

$$\begin{cases} \| CX'_{n+1.000} - DX'_n - EX'_{n-k} - f(n, X'_{n-k}) \| \leq \epsilon \Psi_n, & n \geq 0.00, \\ \| X'_n - \Phi_n \| \leq \epsilon \Psi, & k \in J. \end{cases} \quad (3)$$

$$\begin{cases} \| CX'_{n+1.000} - DX'_n - EX'_{n-k} - f(n, X'_{n-k}) \| \leq \epsilon \varphi_n, & n \geq 0.00, \\ \| X'_n - \Phi_n \| \leq \epsilon \psi_k, & k \in J. \end{cases} \quad (4)$$

Definition 3

The stability of system (2) is denoted as Hyers-Ulam-Rassias stability, if for any ϵ-approximate solution (3) of (2), there exists a corresponding exact solution X_n of (2). Furthermore, there is a non-negative real number $K_{L^4,\eta,\psi,\varphi_n}$ such that

$$\| X_n - X'_n \| \leq K_{L^4,\eta,\psi,\varphi_n} \epsilon(\varphi_n + \psi), \quad \text{for all} \quad n \in j.$$

Definition 4

The stability of system (2) is denoted as stable in terms of generalized Hyers-Ulam-Rassias, if for any ϵ-approximate solution (4) of (2), there exists a corresponding exact solution X_n of (2). Furthermore, there is a non-negative real number $L_{K,L^2,\zeta_f,\lambda_\varphi}$ such that

$$\| X_n - X'_n \| \leq L_{K,L^2,\zeta_f,\lambda_\varphi} \epsilon(\varphi_n + \Psi_{k+1}), \quad n \in I.$$

Lemma 5 (see Samoilenko and Perestyuk (1995)

For any $n \geq 0.00$ with

$$\| U_n \| \leq a_n + \sum_{i=0.00}^{n} Q_i U_i + \sum_{0.00 \leq n_k \leq n} \gamma_k U_{n_k - 1.000}, \quad n \geq 0.00,$$

then we have,

$$\mathcal{U}_n \leq a_n (1.000 + \gamma_k)^k \exp(\sum_{i=0.00}^{n} Q_i), \quad where \quad k \in \mathcal{J}.$$

Remark 6

If we replace γ_k by γ_{k_n} then

$$\mathbb{Y}_n \leq c \prod_{n_k=0.00}^{n} (1.000+\gamma_{k_n})\exp(\sum_{i=0.00}^{n} Q_i) \quad for \quad n \geq 0.00.$$

Stability in terms of Hyers-Ulam-Rassias on Bounded Discrete Interval

For investigating the stability in terms of Hyers-Ulam-Rassias of (2) on discrete bounded interval, let have some conditions:

Λ_1: The system $C\mathrm{X}_{n+1.000} = D\mathrm{X}_n + E\mathrm{X}_{n-k}$ is well modeled.

Λ_2: $P: \mathrm{I} \times \mathbb{X} \to \mathcal{X}$ holds the Caratheodry condition

$$\| \mathrm{P}(t,\propto) - P(t,\propto') \| \leq \mathrm{K} \| \propto - \propto' \|,$$

for $\mathbb{K} \geq 0.00$, and $\forall \propto, \propto' \in \mathrm{B}(\mathbb{J}, X)$.

Λ_3: There exists non-decreasing $\varphi_n \in \mathbb{B}(I,X)$ with a constant $\epsilon > 0.00$ such that

$$\sum_{r=k+1.000}^{n} \| f_{i-k} \| \leq \epsilon(\varphi_n + \Psi) \quad for \quad each \quad n \in I.$$

Λ_4: $\sum_{i=k+1.000}^{n} (E + K_i) \leq \eta_\varphi \Psi_n$.

Λ_5: $\sup \| C \| = \| M \| = M$.

Theorem 7

If $\Lambda_1 - \Lambda_5$ hold, then (2) is stable in terms of Hyers-Ulam-Rassias on discrete bounded interval.

Proof: Now the solution of an equation

$$\begin{cases} C\mathrm{X}_{n+1.000} = \mathbb{D}\mathrm{X}_n + \mathcal{E}\mathrm{X}_{n-l} + \mathbb{F}(n, \mathrm{X}_{n-l}), & n \geq 0.00, \quad l \geq 0.00, \\ \mathrm{X}_n = \Psi_n, & 0.00 \geq n \geq -l, \end{cases}$$

is

$$\mathrm{X}_n = \mathbb{D}^n C^{-n} \Psi_{0.00} + D^{n-1.000} C^{-n} \sum_{i=0.00}^{l} D^{-i} C^i (E\Psi_{i-l} + \mathbb{F}(i, \Psi_{i-l}))$$
$$+ D^{n-1.000} C^{-n} \sum_{i=l+1.000}^{n} D^{-i} C^i \big(E\mathrm{X}_{i-l} + \mathrm{F}(i, \mathrm{X}_{i-l})\big),$$

while

$$\begin{cases} C\mathbb{X}'_{n+1.000} = D\mathbb{X}'_n + \mathrm{E}\mathbb{X}'_{n-l} + \mathrm{F}(n, \mathbb{X}'_{n-l}) + \mathrm{f}_n, & n \geq 0.00, \quad l \geq 0.00, \\ \mathbb{X}'_n = \Phi_n, & -l \leq n \leq 0.00, \end{cases}$$

is satisfied by

$$\mathbb{X}'_n = D^n C^{-n} \Psi_{0.00} + \mathcal{F}(i, \Psi_{i-l}) + D^{n-1.000} C^{-n} \sum_{i=0.00}^{l} D^{-i} C^i (E\Psi_{i-l}$$
$$+ D^{n-1.000} C^{-n} \sum_{i=l+1.000}^{n} D^{-i} C^i (E\mathbb{X}'_{i-l} + \mathrm{F}(i, \mathbb{X}'_{i-l}) + \mathrm{f}_{i-l}).$$

Now, we have,

$$\| X'_n - D^n C^{-n}\phi_{0.00} - D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l} + \mathrm{F}(i, X'_{i-l})) + \mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=k+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} \|$$

$$=\| \mathbb{F}(i,\phi_{i-l}))$$

$$+D^{n}C^{-n}\phi_{0.00} + D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}$$

$$+D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + F(i, X'_{i-l}) + \mathrm{f}_{i-l})$$

$$-D^{n}C^{-n}\phi_{0.00} - D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l} + \mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + F(i, X'_{i-l})) \|$$

$$=\| D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}f_{i-l} \|$$

$$\leq L^4 \sum_{i=l+1.000}^{n} \| \mathrm{f}_{i-l} \|$$

$$\leq L^4\epsilon(\varphi_n + \Psi),$$

now

$$\| X'_n - X_n \| = \| X'_n - D^n C^{-n}\phi_{0.00}$$

$$- D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l} + \mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l} + \mathcal{F}(i, X_{i-l})) \|$$

$$=\| X'_n - D^n C^{-n}\phi_{0.00} - D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l} + \mathbb{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l} + \mathcal{F}(i, X_{i-l}))$$

$$+D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + F(i, X'_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + F(i, X'_{i-l})) \|$$

$$=\| X'_n - D^n C^{-n}\phi_{0.00} - D^{n-1.000}C^{-n}\sum_{i=0.00}^{k} D^{-i}C^{i}(E\phi_{i-l} + \mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+F(i,X'_{i-l}))$$
$$+D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+F(i,X'_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l}+F(i,X_{i-l}))\parallel$$
$$\leq\parallel X'_{n}-D^{n}C^{-n}\phi_{0.00}-D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}+\mathrm{F}(i,\phi_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathrm{F}(i,X'_{i-l}))\parallel$$
$$+\parallel D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathrm{F}(i,X'_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l}+F(i,X_{i-l}))\parallel$$
$$\parallel X'_{n}-X_{n}\parallel\leq M^{4}\epsilon(\varphi_{n}+\Psi)+$$
$$\parallel D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathrm{F}(i,X'_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l}+F(i,X_{i-l}))\parallel$$
$$\leq M^{4}\epsilon(\varphi_{n}+\Psi)+\parallel D^{n-1.000}\parallel\parallel C^{-n}\parallel\sum_{i=l+1.000}^{n}\parallel D^{-i}\parallel\parallel C^{i}\parallel($$
$$\parallel EX'_{i-l}-EX_{i-l}\parallel$$
$$+\parallel \mathbb{F}(i,X'_{i-l})-F(i,X_{i-k})\parallel)$$
$$\leq M^{4}\epsilon(\varphi_{n}+\Psi)+M^{4}\sum_{i=l+1.000}^{n}(\mathbb{E}\parallel X'_{i-l}-\mathrm{X}_{i-l}\parallel+K_{i}$$
$$\parallel X'_{i-l}-X_{i-l}\parallel)$$
$$=M^{4}\epsilon(\varphi_{n}+\Psi)+M^{4}\sum_{i=l+1.000}^{n}(E+K_{i})\parallel X'_{i-l}-X_{i-l}\parallel$$
$$\leq M^{4}\epsilon(\varphi_{n}+\Psi)\exp(M^{4}\sum_{i=l+1.000}^{n}(E+K_{i}))$$
$$\leq M^{4}\epsilon(\varphi_{n}+\Psi)\exp(M^{4}\eta_{\varphi}\psi_{n})$$
$$=K_{M^{4},\eta,\psi,\varphi_{n}}\epsilon(\varphi_{n}+\Psi).$$

Where $K_{M^{4},\eta,\psi,\varphi_{n}}=L^{4}\exp(L^{4}\eta_{\varphi}\psi_{n})$. Hence, (2) is stable in terms of Hyers-Ulam-Rassias on bounded discrete interval.

Stability in terms of Hyers-Ulam-Rassias on Discrete Unbounded Interval

For determining the stability in terms of Hyers-Ulam-Rassias on discrete unbounded interval. We need some assumptions:

A_1: The linear system $AX_{n+1.000} = DX_n + EX_{n-k}$ is well modeled.

A_2 : The continuous map $\mathcal{U}: Z_+ \times X \to X$ satisfies the Caratheodary condition

$$\| U(n,\Omega) - U(n,\Omega') \| \leq K \| \Omega - \Omega' \| , \quad \mathbb{K} \geq 0.00,$$

for every $n \in \mathbb{Z}_+ \ \Omega, \Omega' \in X$.

A_3: $\| D^{n-1.000-i} \| \leq K e^{\omega(n-i)}$.

Λ_4: $\| f_{i-k} \| \leq \epsilon \varphi_{i-k}$.

Λ_5: $\sum_{i=k+1.000}^{n} (E + K_i) \leq \omega_f + \zeta_f$

Λ_6: $\sum_{i=k+1.000}^{n} e^{\omega(n-i)+KL^2\omega_f n} \varphi_{i-k} \leq \lambda_\varphi(\varphi_n + \Psi)$.

Theorem 8

If Λ_1-Λ_6 are satisfied, then (2) is Hyer-Ulam-Rassias stable on unbounded discrete interval.

Proof: Since the exact solution of the equation,

$$\begin{cases} CX_{n+1.000} = DX_n + \mathcal{E}X_{n-l} + F(n, X_{n-l}), & n \geq 0.00, \quad l \geq 0.00, \\ X_n = \psi_n, & -l \leq n \leq 0.00, \end{cases}$$

is

$$X_n = \mathcal{D}^n C^{-n}\psi_{0.00} + \mathbb{F}(i,\psi_{i-l})) + \mathcal{D}^{n-1.000}C^{-n}\sum_{i=0.00}^{l} \mathcal{D}^{-i}C^i(\mathbb{E}\psi_{i-l}$$
$$+\mathcal{D}^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} \mathcal{D}^{-i}C^i(\mathbb{E}X_{i-l} + \mathbb{F}(i, X_{i-l})).$$

Let X be the ϵ −approximate solution of 2.1.000, then for a sequence f_n, with $\| f_n \| \leq \epsilon$ we have

$$\begin{cases} CX'_{n+1.000} = \mathcal{D}X'_n + EX'_{n-l} + F(n, X'_{n-l}) + f_n, & n \geq 0.00, \quad l \geq 0.00, \\ X'_n = \psi_n, & -l \leq n \leq 0.00, \end{cases}$$

and

$$X'_n = \mathcal{D}^n C^{-n}\psi_{0.00} + F(i,\psi_{i-l})) + \mathcal{D}^{n-1.000}C^{-n}\sum_{i=0.00}^{l} \mathcal{D}^{-i}C^i(\mathbb{E}\psi_{i-l}$$
$$+\mathcal{D}^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} \mathcal{D}^{-i}C^i(\mathbb{N}X'_{i-l} + f_{i-l} + F(i, X'_{i-l})).$$

Now,

$$\| X'_n - D^n C^{-n}\phi_{0.00} - D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^i(E\phi_{i-l} + F(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+F(i,X'_{i-l}))\parallel$$

$$=\parallel D^{n}C^{-n}\phi_{0.00}$$

$$+D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}$$

$$+D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathcal{F}(i,X'_{i-l})+f_{i-l}$$

$$+\mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n}C^{-n}\phi_{0.00}-D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}+\mathcal{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathbb{F}(i,X'_{i-l}))\parallel$$

$$=\parallel D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}f_{i-l}\parallel$$

$$\leq\sum_{i=l+1.000}^{n}\parallel D^{n-1.000-i}\parallel\parallel C^{-n+1.000}\parallel\parallel f_{i-l}\parallel$$

$$\leq KL^{2}\sum_{i=l+1.000}^{n} e^{\omega(n-i)}\epsilon\varphi_{i-l},$$

now

$$\parallel X'_{n}-X_{n}\parallel=\parallel X'_{n}-D^{n}C^{-n}\phi_{0.00}$$

$$-D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}+\mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l}+\mathrm{F}(i,X_{i-l}))\parallel$$

$$=\parallel X'_{n}-D^{n}C^{-n}\phi_{0.00}$$

$$-D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}+\mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l}+\mathbb{F}(i,X_{l-k}))$$

$$+D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathbb{F}(i,X'_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l}+\mathrm{F}(i,X'_{i-l}))\parallel$$

$$=\parallel X'_{n}-D^{n}C^{-n}\phi_{0.00}-D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l}$$

$$+\mathrm{F}(i,\phi_{i-l}))$$

$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + \mathrm{F}(i,X'_{i-l}))$$
$$+D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + F(i,X'_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l} + F(i,X_{i-l})) \parallel$$
$$\leq \parallel X'_{n} - D^{n}C^{-n}\phi_{0.00} - D^{n-1.000}C^{-n}\sum_{i=0.00}^{l} D^{-i}C^{i}(E\phi_{i-l} + \mathrm{F}(i,\phi_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + \mathcal{F}(i,X'_{i-l})) \parallel$$
$$+\parallel D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + \mathbb{F}(i,X'_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l} + \mathrm{F}F(i,X_{i-l})) \parallel$$
$$\parallel X'_{n} - X_{n} \parallel \leq KL^{2}\epsilon\sum_{i=l+1.000}^{n} e^{\omega(n-i)}\varphi_{i-l}$$
$$+\parallel D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX'_{i-l} + F(i,X'_{i-l}))$$
$$-D^{n-1.000}C^{-n}\sum_{i=l+1.000}^{n} D^{-i}C^{i}(EX_{i-l} + \mathrm{F}(i,X_{i-l})) \parallel$$
$$\leq KL^{2}\epsilon\sum_{i=l+1.000}^{n} e^{\omega(n-i)}\varphi_{i-l}$$
$$+\sum_{i=l+1.000}^{n} \parallel D^{n-1.000-i} \parallel\parallel C^{-n+i} \parallel (\parallel EX'_{i-l} - EX_{i-l} \parallel + \parallel \mathcal{F}(i,X'_{i-l}) - \mathrm{F}(i,X_{i-l}) \parallel)$$
$$\leq KL^{2}\epsilon\sum_{i=l+1.000}^{n} e^{\omega(n-i)}\varphi_{i-l}$$
$$+KL^{2}\sum_{i=l+1.000}^{n} e^{\omega(n-i)}(E \parallel X'_{i-l} - X_{i-l} \parallel + K_{i} \parallel X'_{i-l} - X_{i-l} \parallel)$$
$$= e^{\omega n}[KL^{2}\epsilon\sum_{i=l+1.000}^{n} e^{-\omega i}\varphi_{i-l} + KL^{2}\sum_{i=l+1.000}^{n} e^{-\omega i}(E + K_{i}) \parallel X'_{i-l} - X_{i-l} \parallel]$$
$$\parallel e^{\omega n}X'_{n} - e^{-\omega n}X_{n} \parallel \leq KL^{2}\epsilon\sum_{i=l+1.000}^{n} e^{-\omega i}\varphi_{i-l}$$

$$+KL^2 \sum_{i=l+1.000}^{n} (E+K_i) \parallel e^{-\omega i}X'_{i-l} - e^{-\omega i}X_{i-l} \parallel$$

$$\parallel \overline{X'_n} - \overline{X_n} \parallel \leq KL^2\epsilon \sum_{i=l+1.000}^{n} e^{-\omega i}\varphi_{i-l}$$

$$+KL^2 \sum_{i=l+1.000}^{n} (E+K_i) \parallel \overline{X'_{i-l}} - \overline{X_{i-l}} \parallel$$

$$\leq KL^2\epsilon \sum_{i=l+1.000}^{n} e^{-\omega i}\varphi_{i-l}\exp(KL^2 \sum_{n=l+1.000}^{n} (E+K_i))$$

$$\leq KL^2\epsilon \sum_{i=l+1.000}^{n} \varphi_{i-l}e^{KL^2(\omega_f n+\zeta_f)}$$

$$= KL^2\epsilon \sum_{i=l+1.000}^{n} e^{-\omega i+KL^2(\omega_f n+\zeta_f)}\varphi_{i-l}$$

$$= KL^2\epsilon e^{KL^2\zeta_f}\epsilon \sum_{i=l+1.000}^{n} e^{-\omega i+KL^2\omega_f n}\varphi_{i-l}$$

$$\parallel X'_n - X_n \parallel = KL^2 e^{KL^2\zeta_f}\epsilon \sum_{i=l+1.000}^{n} e^{\omega(n-i)+KL^2\omega_f n}\varphi_{i-l}$$

$$\leq KL^2 e^{KL^2\zeta_f}\epsilon\lambda_\varphi(\varphi_n + \Psi)$$

$$= L_{K,L^2,\zeta_f,\lambda_\varphi}\epsilon(\varphi_n + \Psi),$$

where $L_{K,L^2,\zeta_f,\lambda_\varphi} = KL^2 e^{KL^2\zeta_f}\lambda_\varphi$. Thus, (2) is Hyer-Ulam-Rassias stable on unbounded discrete interval.

Generalized Hyers-Ulam-Rassias stability

To described about Generalized Hyers-Ulam-Rassias stability, we needed some assumptions:

A_1: The linear system $A\mathrm{X}_{n+1} = D\mathrm{X}_n + E\mathrm{X}_{n-k}$ is well modeled.

A_2: The continuous function A satisfies the Caratheodary condition

$$\parallel \mathbb{A}(n,\Xi) - \mathrm{A}(n,\Xi') \parallel \leq K_n \parallel \Xi - \Xi' \parallel, \quad K \geq 0.00,$$

for every $n \in \mathbb{Z}_+ \ \Xi, \Xi' \in X$.

A_3: Also,

$$\sum_{i=k+1}^{n} e^{-\omega i+(KL^2\sum_{n=k+1}^{n}(E+K_i))}\varphi_{i-k} \leq \rho(\varphi_n + \Psi_{l+1}).$$

A_4: $\parallel M^{n-1-i} \parallel \leq Ke^{\omega(n-i)}$.

Theorem 9

If A_1 - A_4 are satisfied, then (5) is generalized Hyers-Ulam-

Rassias stable.

Proof: The only solution of the given equation

$$\begin{cases} \mathrm{C}\mathrm{X}_{n+1.000} = \mathbb{D}\mathbb{X}_n + \mathrm{E}\mathrm{X}_{n-l} + F(n, \mathcal{X}_{n-l}), & 0.00 \leq n, \quad 0.00 \leq l, \\ \mathcal{X}_n = \Phi_n, \quad -l \leq n \leq 0.00, \end{cases} \tag{5}$$

is

$$\mathcal{X}_n = \mathcal{D}^n \mathcal{C}^{-n} \Phi_{0.00} + \mathcal{D}^{n-1.000} \mathcal{C}^{-n} \sum_{j=0.00}^{l} \mathcal{D}^{-j} \mathcal{C}^{j} (\mathcal{E}\Phi_{j-l} + \mathrm{F}(l, \Phi_{j-l}))$$

$$+\mathcal{D}^{n-1.000} \mathcal{C}^{-n} \sum_{j=l+1.000}^{n} \mathcal{D}^{-j} \mathcal{C}^{j} (\mathcal{E}\mathcal{X}_{j-l} + \mathrm{F}(j, \mathcal{X}_{j-l})).$$

From Theorem (8)

$$\| \overline{X'_n} - \overline{X_n} \| \leq KL^2 \epsilon \sum_{i=l+1.000}^{n} e^{-\omega i} \varphi_{i-l} \exp(KL^2 \sum_{n=l+1.000}^{n} (E + K_i))$$

$$= KL^2 \epsilon \sum_{i=l+1.000}^{n} e^{-\omega i} \varphi_{i-l} e^{(KL^2 \sum_{n=l+1.000}^{n} (E + K_i))}$$

$$= KL^2 \epsilon \sum_{i=l+1.000}^{n} e^{-\omega i + (KL^2 \sum_{n=l+1.000}^{n} (E + K_i))} \varphi_{i-l}$$

$$\| X'_n - X_n \| = KL^2 \epsilon \sum_{i=lo+1.000}^{n} e^{\omega(n-i) + (KL^2 \sum_{n=l+1.000}^{n} (E + K_i))} \varphi_{i-l}$$

$$\leq KL^2 \epsilon \rho (\varphi_n + \Psi_{k+1.000})$$

$$= C_{K,L^2,\rho} \epsilon (\varphi_n + \Psi_{l+1.000}),$$

where $C_{K,L^2,\rho} = KL^2 \rho$. So, system (5) is stable in terms of generalized Hyers-Ulam-Rassias.

Conclusion

In this paper, we studied the Ulam's type stability results of a non-singular and autonomous difference equation with delay. Our work provides assurance regarding the existence of an exact solution near to the approximate solution. Indeed, our findings hold particular significance in situations where obtaining an exact solution is notably challenging, making them valuable in fields like approximation theory. Ulam stability serves a crucial role in the approximation of solutions, particularly in situations where obtaining exact solutions presents challenges. Our future aim is to apply these findings to real-world systems, particularly in the context of mathematical models of the brain.

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