Enhanced Laplace Adomian Decomposition Method for Nonlinear Volterra Integral Equation

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Abstract

The Adomian Decomposition Method (ADM) can be directly applied with constant or variable coefficients without using linearization, destruction or some other unpreferable assumptions. The ADM is found rapidly converging on several type of ordinary and partial differential equations. Though, some valuable and significant modification in ADM like Laplace Adomian Decomposition Method (LADM) and Modified Laplace Adomian Decomposition Method (MLADM) were introduced by researchers. Despite better performance, the effectiveness, weaknesses and inconsistencies of traditional and modification in ADM need to be explored. Moreover, the performance and efficiency of Laplace based ADM need to be further improved. Accordingly, the strength of two existing modifications LADM and MLADM in ADM is integrated and a new technique named Enhanced Laplace Adomian Decomposition Method (ELADM) is introduced in this paper. Some illustrative examples are provided to analyze the working of proposed ELADM, LADM and MLADM techniques where the suggested scheme ELADM has proved accurate findings. The obtained results are graphically presented and are discussed. The convergence of ELADM technique is proved for solving nonlinear Volterra integral equation of second kind. The overall absolute error obtained acknowledges that the solutions by the proposed ELADM technique are very much similar to the exact solution. The suggested ELADM approach is thus easy to adopt, and the precision of the solution is clear.

Keywords: Numerical Laplace Transform Method; Volterra Integral Equations; Adomian Decomposition Method; Newton Raphson Formula

Introduction

The Nonlinear partial and ordinary differential equations and, sometimes, integral or integro differential equations may explain the majority of real-world phenomena. A strong approach for solving nonlinear differential equations was proposed by George Adomian (Adomian, 1988). Since then, this procedure has been referred to as the process of Adomian Decomposition Method. The primary benefit of this technique is that it can be directly applied with constant or variable coefficients to all types of homogeneous or inhomogeneous equation (Achouri & Omrani, 2009). Without using linearization, destruction or

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some other unpreferable assumptions that may alter the physical behavior of the model, the ADM solves the problems explicitly and in an uncomplicated way (Jiao et al., 2002). Another major gain is that the approach is capable of significantly decreasing the measurement of computational work. The rapid convergence of Adomian decomposition method has been investigated by (Abbaoui & Cherruault, 1994; Babolian & Biazar, 2002). Several linear and nonlinear ODE's , PDE's, Volterra integral equation, Volterra integro differential equation, Fredholm integral equation are solved by using Adomian decomposition method (Adomian, 1988); Biazar & Shafiof, 2007; Babolian & Mordad, 2011); Wazwaz, 1998; Nhawu et al., 2016). Many researchers modified classical ADM through different aspects (Hussain, 2019; Xie, 2013; Hamoud & Ghadle, 2019).

In contrast with the conventional procedure of decomposition, the ADM affiliations with the Laplace transform borrow less work. This was first proposed by (Khuri, 2001). The Laplace transform is an integral transform observed by Pierre-Simon Laplace L and is a strong and very valuable method for solving ordinary and partial differential equations which transform the original differential equation into an elementary algebraic equation. The Laplace Adomian Decomposition Method's primary benefit is the independence of parameters, small or large (Hosseinzadeh et al., 2010). To achieve the exact solution of nonlinear equations, this procedure is used, but for inhomogeneous differential equations, it generates a noise term. Adomian and Rach recently presented the so-called "noise terms" phenomenon. The terms are described as the identical terms with adverse signs that appear in the components of the series solution of $u(x)$.

In (Khan et al., 2012), it is assumed that if terms in the u_0 component are cancelled by terms in the u_1, component, even if u_1, contain additional terms, the remaining non-cancelled u_0 terms provide the exact solution. It was indicated in (Adomian & Rach, 1992) that the noise term appears always for inhomogeneous equations. In (Wazwaz & Mehanna, 2010) the author investigated the combined form of LADM for analytical treatment of the nonlinear singular integral equation describing heat transfer. Laplace Adomian decomposition method applied to solve Burgers equation to prove their convergence in (Naghipour & Manafian, 2015). A computational technique applied for solving linear and nonlinear Volterra integral equation of weakly kernels (Hendi, 2011). To find the analytical solution of the linear and nonlinear systems of partial differential equations, a numerical Laplace transform algorithm based on the Adomian Decomposition Method is presented in (Fadaei, 2011; Hamoud & Ghadle, 2017; Hussain, 2019). Some authors also improve and modify the technique from various aspects (Hamoud & Ghadle, 2017; Hussain, 2019). Therefore, there are abundant applications where many researchers use the Laplace Adomian Decomposition process (Hussain & Khan, 2010; Khan & Faraz, 2011; Heris, 2012; Olubanwo et al., 2015; Hamoud & Ghadle, 2017; Rani & Mishra, 2019). Our work is inspired by their work, and we have tried to further improve the Modified Laplace Adomian Decomposition Method technique.

In the present paper, our goal of research is to develop an enhancement in LADM named Enhanced Laplace Adomian Decomposition Method for finding the approximate solution of nonlinear Volterra integral equations. In several research areas, such as the population dynamics of epidemics and semi-conductor systems, Nonlinear Volterra integral equations emerge (Wazwaz, 2011). Many researchers have recently inquired about the solution to this problem. Existing methods are presented to solve this kind of equations (Nhawu et al., 2016; Hussain, 2019; Duan et al., 2012; Almousa, 2020).

In section 2, a brief discussion for the Enhanced Laplace Adomian Decomposition method particularly on Nonlinear Volterra integral equation of the second kind is presented. In Section 3, applications of this method and numerical results by LADM, MLADM and ELADM are illustrated and discussed. Section 4 ends this paper with a brief conclusion.

Analysis of Enhanced Laplace Adomian Decomposition Method (ELADM) on Nonlinear Volterra Integral Equation

Previously Newton Raphson formula is used instead of Adomian polynomial (Rani & Mishra, 2018) and in another modification (Hussain, & Khan, 2010), the Laplace Adomian decomposition method is done by splitting the value of u_0 in to two terms $u_1 + u_2$. Accordingly, in this research we integrate these two modifications as a single method called Enhanced Laplace Adomian Decomposition Method (ELADM) to obtain the approximate solution of nonlinear Volterra Integral equations.

Consider the non-linear Volterra integral equation with difference kernel, i.e.

$$u(x) = f(x) + \int_0^x k(x-t)F\big(u(t)\big)dt \tag{1}$$

where $f(x)$ is a known real valued function, $k\,(x,t) = k\,(x-t)$ and $F\big(u(x)\big)$ is $u(x)$'s nonlinear function.

On both sides of (1), apply Laplace Transform. Use the linear property afterwards and Laplace transformation theorem of Convolution, we have:

$$L[u(x)]{=}L[f(x)]{+}L[k(x-t)]L[F(u(x))] \tag{2}$$

The approach requires the approximation of the (1) solution as an infinite series provided by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (3)$$

The nonlinear expression $F(u(x))$ is however disintegrated as:

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \quad (4)$$

where, $A_n(x)$ are the so-called Adomian polynomials that can be calculated using the following formula:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n} \lambda^i \left(u_i - \frac{F(u_i)}{F'(u_i)}\right)\right)\right] \lambda = 0,\ n \geq 0 \quad (5)$$

By putting Equations (4) and (3) into (2), we get:

$$L[\sum_{n=0}^{\infty} u_n(x)] = L[f(x)] + L[k(x-t)]L[\sum_{n=0}^{\infty} A_n(x)] \quad (6)$$

Using the linearity property of the Laplace transform, we obtained:

$$\sum_{n=0}^{\infty} L[u_n(x)] = L[f(x)] + L[k(x-t)] \sum_{n=0}^{\infty} L\,[A_n(x)] \quad (7)$$

To find the terms $u_0(x),\ u_1(x), u_2(x), u_3(x)$ ... We have the following iterative scheme of infinite series, matching both sides of (7):

$$L[u_0(x)] = L[f(x)] \quad (8)$$

Generally, the relation shows:

$$L[u_{n+1}(x)] = L[k(x-t)]L[A_n(x)] \quad n \geq 0 \quad (9)$$

Apply the inverse Laplace transformation to (8) and (9), we get:

$$u_0(x) = L^{-1}\big[L[f(x)]\big] \quad (10)$$

$$u_{n+1}(x) = L^{-1}\big[L[k(x-t)]L[A_n(x)]\big] \quad n \geq 0 \quad (11)$$

We use Laplace now, first of all, to change the terms on the right side of Equation (10) we gain the values of $u_0(x)$,$u_1(x)$, ... $u_n(x)$ respectively by applying the inverse Laplace transformation.

We consider that $f(x)$ can be split into the sum of two terms, namely $f_0(x)$ and $f_1(x)$ to apply this modification, so we get:

$$f(x) = f_0(x) + f_1(x) \quad (12)$$

Under this consideration, we suggest a slight variation only in the components u_0, u_1. The variation we suggest is that only the part $f_0(x)$ be assigned to the u_0, whereas the remaining part $f_1(x)$ be merged with the other terms described in Equation (11) to define u_1. In consideration of these recommendations, the modified recursive algorithm is formulated as follows:

$$u_0(x) = L^{-1}\big[L[f_0(x)]\big] \quad (13)$$

$$u_1(x) = f_1(x) + L^{-1}\big[L[k(x-t)]L[A_0(x)]\big] \quad (14)$$

$$u_{n+1}(x) = L^{-1}\big[L[k(x-t)]L[A_n(x)]\big] \quad (15)$$

Experimental Results on Several Modifications in Adomian Decomposition Method

This section elaborates and explains the effectiveness and generalization of proposed ELADM by comparing its results with some modifications in ADM. Three examples of Nonlinear Volterra Integral equation are solved on Laplace Adomian Decomposition method

(LADM), Modified Laplace decomposition (MLADM) and Enhanced Laplace Adomian decomposition method (ELADM). The results are compared with exact solution. Moreover, the Absolute Error value for each example is also calculated. The results are presented in the form of tables and graphs. In order to validate the proposed ELADM for solving the nonlinear Volterra integral equation, three different examples are considered.

Example 1

Solve the following nonlinear Volterra integral equation (Wazwaz, 2015):

$$u(x) = x^2 - \frac{1}{30}x^6 + \int_0^x (x-t)u^2(t)\,dt \tag{16}$$

having exact solution $u(x) = x^2$.

Solution

Case I: LADM

In this case, we will solve Equation (16) using Laplace Adomian Decomposition Method based on Newton Raphson formula:

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \tag{17}$$

Implementing Laplace transform on both sides of Equation (16) and by using the linearity property, we have:

$$\mathscr{L}[u(x)] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] + \mathcal{L}[x]\mathcal{L}[u^2(x)] \tag{18}$$

The approach assumes that the functional series solution is $u(x)$:

$$\mathscr{L}\left[\sum_{n=0}^{\infty} u_n(x)\right] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] + \frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(x)\right] \tag{19}$$

Using the formula given in Equation (5), the nonlinear expression $F(u(x))$ = $u^2(x)$ is broken down. The following are the terms of modified Adomian polynomials:

$A_0 = \left(\frac{1}{2}\right)^2 u_0^2$, $A_1 = \left(\frac{1}{2}\right)^2 (u_0u_1)$, $A_2 = \left(\frac{1}{2}\right)^2 (2u_0u_2 + u_1^2)$, $A_3 = \left(\frac{1}{2}\right)^2 (2u_0u_3 + 2u_1u_2)$.

Comparing both sides of Equation (19), gives the continual algorithm:

$$\mathcal{L}[u_0(x)] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] \tag{20}$$

In general,

$$\mathcal{L}[u_{n+1}(x)] = \frac{1}{s^2}\mathcal{L}[A_n(x)] \quad n \geq 0 \tag{21}$$

The translation of inverse Laplace to the above iterative steps means:

$$u_0(x) = x^2 - \frac{1}{30}x^6 \tag{22}$$

Using general relation, we have:

$$u_1(x) = \frac{x^6}{120} + \frac{x^{14}}{655200} - \frac{x^{10}}{5400} \quad (23)$$

$$u_2(x) = \frac{x^{10}}{43200} + \frac{227x^{18}}{36088415584} - \frac{x^{14}}{1572480} - \frac{x^{22}}{36324287224} \quad (24)$$

and so forth.

Consequently, the solution comes in the form: $u(x) = x^2 - \frac{x^6}{40} - \frac{7x^{10}}{43200} + \frac{x^{14}}{1123200} + \frac{227x^{18}}{36088415584} - \frac{x^{22}}{36324287224}\ldots.$

Case II: MLADM

The same example is now solved by Modified Laplace Adomian decomposition method. Applying transforming Laplace on both sections of Equation (1):

$$\mathcal{L}[u(x)] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] + \mathcal{L}[x]\mathcal{L}[u^2(x)] \quad (25)$$

The approach assumes that the series function solution $u(x)$ is:

$$\mathcal{L}[\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] + \frac{1}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} A_n(x)] \quad (26)$$

Applying inverse Laplace transformation on both sides of Equation (26):

$$\mathcal{L}^{-1}\left[\mathcal{L}[\sum_{n=0}^{\infty} u_n(x)]\right]=\mathcal{L}^{-1}\left[\mathcal{L}\left[x^2 - \frac{1}{30}x^6\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} A_n(x)]\right] \quad (27)$$

The nonlinear term $F(u(x)) = u^2(x)$ is broken down by utilizing the formula described in Equation (5). Comparing both sides of Equation (27), gives the Modified Laplace algorithm is given below:

$$u_0(x) = x^2 \quad (28)$$

$$u_1(x) = -\frac{1}{30}x^6 + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_0(x)]\right] \quad (29)$$

In general,

$$u_{n+1}(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_n(x)]\right] \quad n \geq 1 \quad (30)$$

From Equation (29) we find the value of $u_1(x)$:

$$u_1(x) = -\frac{1}{30}x^6 + \frac{x^5}{10} \quad (31)$$

By using the general relation, we find the value of $u_2(x)$:

$$u_2(x) = -\frac{x^9}{270} + \frac{x^8}{80} \quad (32)$$

and so forth.

The solution thus takes the form of: $u(x) = x^2 - \frac{x^6}{30} + \frac{x^5}{10} - \frac{x^9}{270} + \frac{x^8}{80}$ ...

Case III: ELADM

In this case we will take the same example and solve this by Enhanced Laplace Adomian decomposition method. Applying transforming Laplace on both sides of Equation (1):

$$\mathscr{L}[u(x)] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] + \mathcal{L}[x]\mathcal{L}[u^2(x)] \quad (33)$$

The approach assumes that the functional series solution is $u(x)$:

$$\mathscr{L}[\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[x^2 - \frac{1}{30}x^6\right] + \frac{1}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} A_n(x)] \quad (34)$$

Applying inverse Laplace transform on both sections of Equation (34) :

$$\mathcal{L}^{-1}\left[\mathcal{L}[\sum_{n=0}^{\infty} u_n(x)]\right]=\mathcal{L}^{-1}\left[\mathcal{L}\left[x^2 - \frac{1}{30}x^6\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} A_n(x)]\right] \quad (35)$$

The nonlinear term $F(u(x)) = u^2(x)$ is broken down by utilizing the algorithm describe by Equation (5). Comparing both sides of Equation (35), gives the Enhanced Laplace algorithm is given below:

$$u_0(x) = x^2 \quad (36)$$

$$u_1(x) = -\frac{1}{30}x^6 + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_0(x)]\right] \quad (37)$$

In general,

$$u_{n+1}(x) = \mathcal{L}^{-\mathbf{1}}\left[\frac{1}{s^2}\mathcal{L}[A_n(x)]\right] \qquad n \geq 1 \quad (38)$$

From Equation (38) we find the value of $u_1(x)$:

$$u_1(x) = -\frac{x^6}{40} \quad (39)$$

Thus by using general relation we have:

$$u_2(x) = -\frac{x^{10}}{14400} \quad (40)$$

$$u_3(x) = \frac{x^{14}}{10483200} \quad (41)$$

and so forth.

Thus the solution takes the form: $u(x) = x^2 - \frac{x^6}{40} - \frac{x^{10}}{14400} + \frac{x^{14}}{10483200} \dots$

In Table 1, we compare the exact solution and approximate solution of LADM based on Newton Raphson formula and MLADM with ELADM for nonlinear Volterra integral equation for example 1. Also we find their absolute error value and we see that the value of ELADM is very close to exact solution than that of LADM and MLADM. By this comparison we conclude that LADM perform better than MLADM, but our proposed ELADM perform better as compared to both LADM and MLADM according to absolute error value. Figure 1 shows the graphical representation of the exact solution and approximate solutions which shows the closeness to the exact solution.

Example 2

Solve the following nonlinear Volterra integral equation:

$$u(x) = 2x - \frac{1}{12}x^4 + 0.25\int_0^x (x-t)u^2(t)\,dt \quad (42)$$

having exact solution $u(x) = 2x$.

Table 1: Comparison of Absolute Error of Different Technique for Example 1.

X	Exact Solution	ELADM	MLADM	LADM	Absolute Error of ELADM	Absolute Error of MLADM	Absolute Error of LADM
0	0	0	0	0	0	0	0
0.05	0.0025	0.00696	2.500×10^{-3}	0.004999	4×10^{-10}	3.07×10^{-8}	1×10^{-9}
0.1	0.01	0.03998	0.0100028	0.029998	2.5×10^{-8}	9.68×10^{-7}	2.5×10^{-8}
0.15	0.0225	0.024223	0.02312212	0.023972	2.97×10^{-7}	7.22×10^{-6}	2.848×10^{-7}
0.2	0.04	0.06869	0.0400677	0.049984	1.60×10^{-6}	2.99×10^{-5}	1.6×10^{-6}
0.25	0.0625	0.06542	0.0627154	0.063497	6.11×10^{-6}	8.92×10^{-5}	6.104×10^{-6}
0.3	0.09	0.09984	0.0902323	0.099834	1.83×10^{-5}	2.25×10^{-4}	1.823×10^{-5}
0.35	0.1225	0.123404	0.123664662	0.123404	4.67×10^{-5}	4.82×10^{-4}	4.711×10^{-5}
0.4	0.16	0.28993	0.1624688	0.159989	1.02×10^{-4}	8.97×10^{-4}	1.024×10^{-4}
0.45	0.2025	0.2023242	0.2042054	0.202324	2.13×10^{-4}	1.67×10^{-3}	2.083×10^{-4}
0.5	0.25	0.31312	0.2535761	0.256122	3.92×10^{-4}	2.7×10^{-3}	3.923×10^{-4}

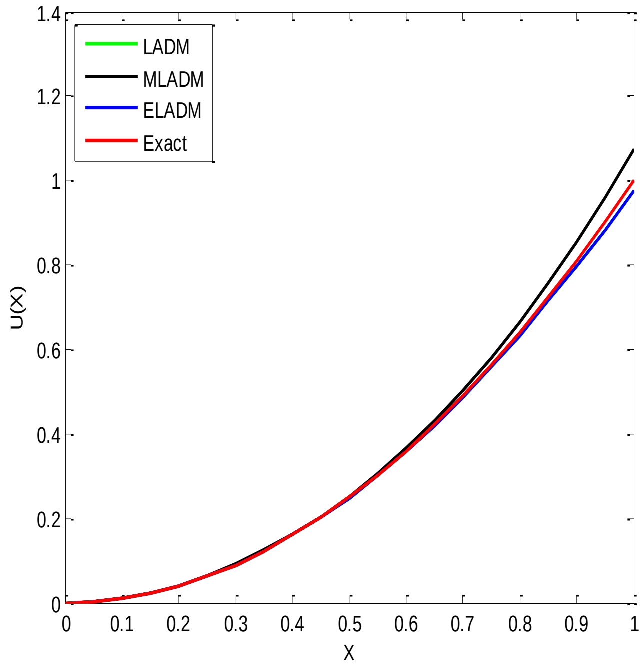

Figure 1: Comparison of Exact Solution and Approximate Solution for Example 1.

Solution

Case I: LADM

In this case, we will solve Equation (42) using Laplace Adomian Decomposition Method based on Newton Raphson formula. On both sides of Equation (42), applying Laplace transform:

$$\mathscr{L}[u(x)] = \mathcal{L}\left[2x - \frac{1}{12}x^4\right] + 0.25\,\mathcal{L}[x]\mathcal{L}[u^2(x)] \tag{43}$$

The approach consider that the series function solution is $u(x)$:

$$\mathscr{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[2x - \frac{1}{12}x^4\right] + \frac{1}{4s^2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)] \qquad 44)$$

The nonlinear expression $F(u(x)) = u^2(x)$ is broken down by utilizing the algorithm given by Equation (5). The continuous algorithm is given by comparing the two sides of Equation (44):

$$u_0(x) = 2x - \frac{1}{12}x^4 \tag{45}$$

In general,

$$\mathcal{L}[u_{n+1}(x)] = \frac{1}{4s^2}\mathcal{L}[A_n(x)] \quad n \geq 0 \tag{46}$$

The translation of inverse Laplace to the above iterative steps means:

$$u_0(x) = 2x - \frac{1}{12}x^4 \tag{47}$$

By using the general relation we get:

$$u_1(x) = \frac{x^{10}}{207360} - \frac{x^7}{2016} + \frac{x^4}{48} \tag{48}$$

$$u_2(x) = -\frac{x^{16}}{4777574400} + \frac{37x^{13}}{905748480} - \frac{11x^{10}}{2903040} + \frac{x^7}{8064} \tag{49}$$

and so on.

Consequently, the solution comes in the form of: $u(x) = 2x - \frac{x^4}{16} - \frac{x^7}{2688} + \frac{x^{10}}{967680} + \frac{37x^{13}}{905748480} - \frac{x^{16}}{4777574400} + \cdots$

Case II: MLADM

The same example is now solved by Modified Laplace Adomian decomposition method:

$$\mathscr{L}[u(x)] = \mathcal{L}\left[2x - \frac{1}{12}x^4\right] + 0.25\,\mathcal{L}[x]\mathcal{L}[u^2(x)] \tag{50}$$

The approach assumes that the series function solution $u(x)$ is:

$$\mathscr{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[2x - \frac{1}{12}x^4\right] + \frac{1}{4s^2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)] \tag{51}$$

Applying the inverse transform of Laplace on both sides of Equation (51):

$$\mathcal{L}^{-1}\left[\mathcal{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)]\right] = \mathcal{L}^{-1}\left[\mathcal{L}\left[2x - \frac{1}{12}x^4\right]\right]\mathcal{L}^{-1}\left[\frac{1}{4s^2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)]\right] \tag{52}$$

If the nonlinear expression $F(u(x)) = u^2(x)$ is decomposed by the algorithm given in (5). The continuous algorithm is given by comparing both sides of Equation (52):

$$u_0(x) = 2x \tag{53}$$

$$u_1(x) = -\frac{1}{12}x^4 + \mathcal{L}^{-1}\left[\frac{1}{4s^2}\mathcal{L}[A_0(x)]\right] \tag{54}$$

In general,

$$u_{n+1}(x) = \mathcal{L}^{-1}\left[\frac{1}{4s^2}\mathcal{L}[A_n(x)]\right] \quad n \geq 1 \tag{55}$$

From the above scheme we find that:

$$u_1(x) = -\frac{x^4}{12} + \frac{x^3}{6} \tag{56}$$

Using the general relation we get:

$$u_2(x) = -\frac{x^6}{144} + \frac{x^5}{60} \tag{57}$$

and so on.

Consequently, the solution comes in the form of: $u(x) = 2x - \frac{x^4}{12} + \frac{x^3}{6} - \frac{x^6}{144} + \frac{x^5}{60} \ldots$

Case III: ELADM

In this case we take the same example and solve this by Enhanced Laplace Adomian decomposition method. On both sides of Equation (42), applying Laplace transform:

$$\mathscr{L}[u(x)] = \mathcal{L}\left[2x - \frac{1}{12}x^4\right] + 0.25\,\mathcal{L}[x]\mathcal{L}[u^2(x)] \tag{58}$$

The approach assumes that the functional series solution is $u(x)$:

$$\mathscr{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[2x - \frac{1}{12}x^4\right] + \frac{1}{4s^2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)] \tag{59}$$

Applying an inverse transform of Laplace on both sides of Equation (59):

$$\mathcal{L}^{-1}\left[\mathcal{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)]\right]=\mathcal{L}^{-1}\left[\mathcal{L}\left[2x - \frac{1}{12}x^4\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{4s^2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)]\right] \tag{60}$$

The nonlinear expression $F(u(x)) = u^2(x)$ is broken down by utilizing the formula presenter by Equation (5). The continuous algorithm gives the relation of both sides of Equation (60):

$$u_0(x) = 2x \tag{61}$$

$$u_1(x) = -\frac{1}{12}x^4 + \mathcal{L}^{-1}\left[\frac{1}{4s^2}\mathcal{L}[A_0(x)]\right] \tag{62}$$

In general,

$$u_{n+1}(x) = \mathcal{L}^{-1}\left[\frac{1}{4s^2}\mathcal{L}[A_n(x)]\right] n \geq 1 \tag{63}$$

Using the above iterative step we get:

$$u_1(x) = -\frac{x^4}{16} \tag{64}$$

$$u_2(x) = -\frac{x^7}{5376} \tag{65}$$

$$u_3(x) = \frac{17x^{10}}{7741440} \tag{66}$$

and so forth.

Hence, the solution comes in the form of: $u(x) = 2x - \frac{x^4}{16} - \frac{x^7}{5376} + \frac{17x^{10}}{7741440} + \cdots$

The comparison of exact solution and approximate solution is given in Table 2. The comparison shows that the absolute error value of ELADM is less than that of LADM based on Newton Raphson formula and MLADM for nonlinear Volterra integral equation of example 2. Here we also noted that the outcome of LADM is best than that of MLADM. However the performance of our technique ELADM is good as compared to both LADM and MLADM. Figure 2 shows the graphical representation of the approximate solution which is very much close to exact solution.

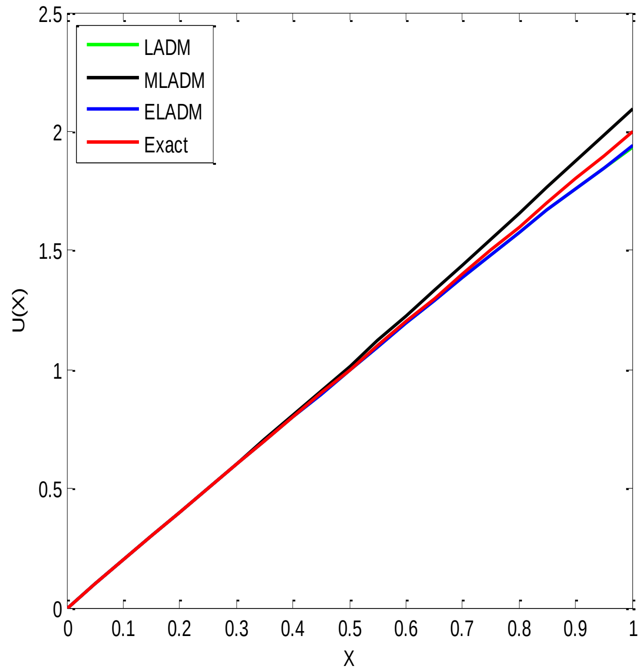

Figure 2: Comparison of Exact Solution and Approximate Solution for Example 2.

Example 3

Solve the following nonlinear Volterra integral equation:

$$u(x) = x^2 + \frac{1}{10}x^5 - \frac{1}{2}\int_0^x u^2(t)\, dt \tag{67}$$

having exact solution $u(x) = x^2$.

Solution

Case I: LADM

In this case, we will solve Equation (67) using Laplace Adomian Decomposition Method. Applying transformation to Laplace on both aspects of Equation (67):

$$\mathscr{L}[u(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] - \frac{1}{2}\mathcal{L}[u^2(x)] \quad 68)$$

The approach assumes that the functional series solution $u(x)$ is:

$$\mathscr{L}[\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] - \frac{1}{2}\mathcal{L}[\sum_{n=0}^{\infty} A_n(x)] \tag{69}$$

Table 2: Comparison of Absolute Error of Different Techniques for Example 2.

X	Exact Solution	ELADM	MLADM	LADM	Absolute Error of ELADM	Absolute Error of MLADM	Absolute Error of LADM
0	0	0	0	0	0	0	0
0.05	0.1	0.0999996	0.10002032	0.1999961	3.913×10^{-7}	2.032×10^{-5}	3.91×10^{-7}
0.1	0.2	0.1999945	0.20028531	0.1999938	6.3×10^{-6}	1.653×10^{-4}	6.3×10^{-6}
0.15	0.3	0.2999741	0.3012159	0.2999745	3.241×10^{-5}	5.229×10^{-4}	3.341×10^{-5}
0.2	0.4	0.4999998	0.40121889	0.4999995	1.002×10^{-4}	1.219×10^{-3}	1.001×10^{-4}
0.25	0.5	0.4997565	0.50233235	0.4997559	2.442×10^{-4}	2.332×10^{-3}	2.443×10^{-4}
0.3	0.6	0.5995409	0.60386045	0.5994937	5.063×10^{-4}	3.905×10^{-3}	5.133×10^{-4}
0.35	0.7	0.6991398	0.70670132	0.6990619	9.401×10^{-4}	5.971×10^{-3}	9.413×10^{-4}
0.4	0.8	0.7984976	0.80867556	0.7983994	1.600×10^{-3}	8.757×10^{-3}	1.601×10^{-3}
0.45	0.9	0.8974442	0.91202024	0.8974362	2.649×10^{-3}	0.012020243	2.643×10^{-3}
0.5	1	0.9961249	1.01603733	0.9960909	3.920×10^{-3}	0.0160433	3.913×10^{-3}

The nonlinear term $F(u(x)) = u^2(x)$ is decomposed by the formula given in Equation (5). The continuous algorithm gives the relation of both section of Equation (69):

$$\mathcal{L}[u_0(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] \quad (70)$$

In general,

$$\mathcal{L}[u_{n+1}(x)] = -\frac{1}{2}\mathcal{L}[A_n(x)] \quad n \geq 1 \quad (71)$$

The translation of inverse Laplace to the above iterative steps means that:

$$u_0(x) = x^2 + \frac{1}{10}x^5 \quad (72)$$

$$u_1(x) = -\frac{x^4}{8} - \frac{x^{10}}{800} - \frac{8x^7}{315} \quad (73)$$

$$u_2(x) = \frac{x^6}{64} + \frac{2101x^{12}}{4435200} + \frac{191x^9}{40320} + \frac{x^{15}}{64000} \quad (74)$$

and so forth.

Consequently, thus the solution comes in the form of: $u(x) = x^2 + \frac{x^5}{10} - \frac{x^4}{8} - \frac{x^{10}}{800} - \frac{8x^7}{315} + \frac{x^6}{64} + \frac{2101x^{12}}{4435200} + \frac{191x^9}{40320} + \frac{x^{15}}{64000}$

Case II: MLADM

The same example is now solved by Modified Laplace Adomian decomposition method. Applying transformation to Laplace on both sides of Equation (67):

$$\mathcal{L}[u(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] - \frac{1}{2}\mathcal{L}[u^2(x)] \quad (75)$$

The approach assumes the series function solution $u(x)$ is:

$$\mathcal{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] - \frac{1}{2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)] \quad (76)$$

Applying an inverse transform of Laplace on both sides of Equation (76):

$$\mathcal{L}^{-1}\left[\mathcal{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)]\right] = \mathcal{L}^{-1}\left[\mathcal{L}\left[x^2 + \frac{1}{10}x^5\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)]\right] \quad (77)$$

The nonlinear expression $F(u(x)) = u^2(x)$ is broken down using the algorithm given by Equation (5). The continuous algorithm makes the comparison of both sides of Equation (77):

$$u_0(x) = x^2 \quad (78)$$

$$u_1(x) = \frac{1}{10}x^5 - \mathcal{L}^{-1}\left[\frac{1}{2}\mathcal{L}[A_0(x)]\right] \quad (79)$$

In general,

$$u_{n+1}(x) = -\mathcal{L}^{-\mathbf{1}}\left[\frac{1}{2}\mathcal{L}[A_n(x)]\right] \quad n \geq 1 \quad (80)$$

$$u_1(x) = \frac{x^5}{10} - x^3 \quad (81)$$

$$u_2(x) = -\frac{7x^6}{20} + \frac{5x^4}{2} \quad (82)$$

Consequently, the solution takes the form of: $u(x) = x^2 + \frac{x^5}{10} - x^3 - \frac{7x^6}{20} + \frac{5x^4}{2}$

Case III: ELADM

In this case we will take the same example and solve this by Enhanced Laplace Adomian decomposition method. Applying transformation to Laplace on both sides of Equation (67):

$$\mathcal{L}[u(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] - \frac{1}{2}\,\mathcal{L}[u^2(x)] \quad (83)$$

The approach assumes that the series function solution is $u(x)$:

$$\mathcal{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)] = \mathcal{L}\left[x^2 + \frac{1}{10}x^5\right] - \frac{1}{2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)] \quad (84)$$

Applying the inverse transform of Laplace on both sides of Equation (84)

$$\mathcal{L}^{-1}\left[\mathcal{L}[\textstyle\sum_{n=0}^{\infty} u_n(x)]\right]=\mathcal{L}^{-1}\left[\mathcal{L}\left[x^2 + \frac{1}{10}x^5\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty} A_n(x)]\right] \quad (85)$$

The nonlinear term $F(u(x)) = u^2(x)$ is decomposed with the formula given by Equation (5). The continuous algorithm makes a comparison of both sides of Equation (85):

$$u_0(x) = x^2 \quad (86)$$

$$u_1(x) = \frac{1}{10}x^5 - \mathcal{L}^{-1}\left[\frac{1}{2}\mathcal{L}[A_0(x)]\right] \quad (87)$$

In general,

$$u_{n+1}(x) = -\mathcal{L}^{-\mathbf{1}}\left[\frac{1}{2}\mathcal{L}[A_n(x)]\right] \quad n \geq 1 \quad (88)$$

$$u_1(x) = \frac{x^5}{10} - \frac{x^4}{8} \quad (89)$$

$$u_2(x) = -\frac{x^7}{80} + \frac{x^6}{64} \quad (90)$$

and so forth.

Therefore, the solution comes in the form of: $u(x) = x^2 + \frac{x^5}{10} - \frac{x^4}{8} - \frac{x^7}{80} + \frac{x^6}{64}$

The particular solution of Laplace Adomian decomposition method based on Newton Raphson formula and Modified Laplace Adomian Decomposition method and the one result by our strategy corresponding to the different x values are show in Table 3 and seen in Figure 3. The absolute error set out in the table acknowledges that the solutions obtained by our technique ELADM are very much similar to the exact solution.

Table 3: Comparison of Absolute Error of Different Techniques for Example 3

X	Exact Solution	ELADM	MLADM	LADM	Absolute Error of ELADM	Absolute Error of MLADM	Absolute Error of LADM
0	0	0	0	0	0	0	0
0.05	0.0025	0.0043023	2.41×10^{-3}	0.0027302	7.67×10^{-7}	1.094×10^{-4}	7.498×10^{-7}
0.1	0.01	0.0295144	9.32×10^{-3}	0.050743	1.173×10^{-5}	7.494×10^{-4}	0.0407338
0.15	0.0225	0.022512	0.02044232	0.022572	5.632×10^{-5}	2.11×10^{-3}	5.5634×10^{-5}
0.2	0.04	0.048334	0.040096	0.3983303	1.68×10^{-4}	3.990×10^{-3}	0.36337503
0.25	0.0625	0.0621125	0.05733203	0.062115	3.934×10^{-4}	5.853×10^{-3}	3.9344×10^{-4}
0.3	0.09	0.0892395	0.08326	0.0924423	7.783×10^{-4}	6.822×10^{-3}	7.6555×10^{-4}
0.35	0.1225	0.1212012	0.1170231	0.1212221	1.352×10^{-3}	5.582×10^{-3}	1.3542×10^{-3}
0.4	0.16	0.158752	0.159604	0.157952	2.133×10^{-3}	4.096×10^{-4}	2.1531×10^{-3}
0.45	0.2025	0.1993035	0.21335883	0.1993847	3.199×10^{-3}	0.0103583	3.2433×10^{-3}
0.5	0.25	0.2469844	0.278913	0.2454645	4.540×10^{-3}	0.0390625	4.6343×10^{-3}

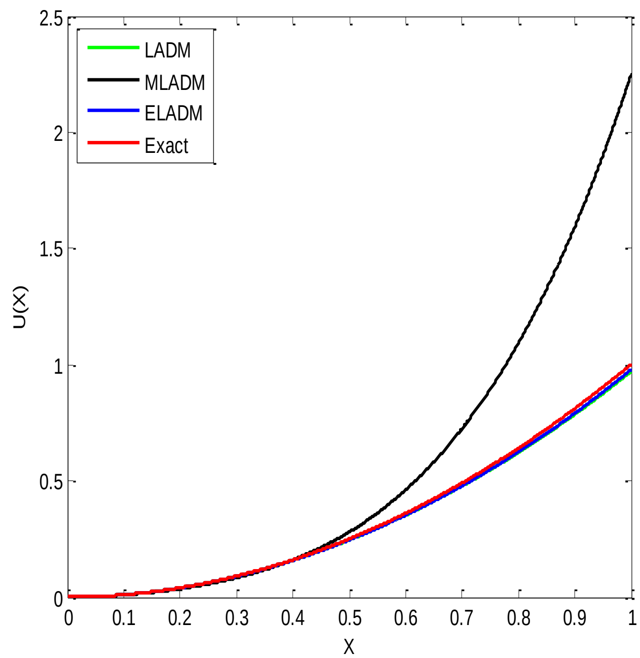

Figure 3: Comparison of Exact Solution and Approximate Solution for Example 3

Conclusion

In this research paper, we have presented the combination of two powerful modifications for solving Nonlinear Volterra integral equations, which are known as Enhanced Laplace Adomian decomposition method. The method that has been proposed in this research is capable of handling a wide class of nonlinear Volterra integral equation. It is noted that ELADM minimize the computational work as compared to existing modifications in ADM like LADM and MLADM. The solution illustrated in the form of tables and figures indicate that the ELADM has a good approximation to the exact solution and has less absolute error as compared to that of LADM and MLADM. It is also observed that while implementing the techniques in solving non-linear Volterra integral equation the LADM outperforms MLADM. However our proposed Enhanced Laplace Adomian Decomposition method performs better as compared to both LADM and MLADM for all considered cases. Therefore we can say that the proposed ELADM is a more generally efficient and effective method for solving non-linear Volterra integral equations. It is also worth mentioning that the feature of the proposed approach is to demonstrate a successful convergence of the solution. Thus the proposed method is easily implemented and manifestly shows the accuracy of solution.

The completion of this research led to an understanding of many topics, such as the nonlinear Volterra integro differential equation, that require further investigation. Despite the better performance, an attempt should be made to increase the consistency of the solution. The definition of linear or nonlinear operators and the use of an alternative transformation are possible areas for this to be explored. The Adomian polynomials could also be examined further. This idea of integrating the strong concepts of individual methods may be used to examine the challenges of convergence and the efficiency of other classical techniques.

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