Unsteady Stokes Flow of a non-Newtonian Fluid Between Two Parallel Porous Plates with Periodic Suction and Injection

Younis Ali[*], Kaleemullah Bhatti[†], Zarqa Bano[‡], Afaque Ahmed Bhutto[§]

Abstract

Unsteady Stokes flow of a non-Newtonian fluid is the time-dependent motion of a fluid where inertial forces are negligible and the fluid exhibits non-Newtonian behaviour, which means that its viscosity is not constant and can depend upon factors such as the rate of strain, time, or fluid history. Non-Newtonian fluids, including viscoelastic and shear-thinning, are used in many applications, particularly in the flow of blood, synovial fluids, and mucus, as well as in industry like polymer extrusion and oil recovery. The current research is based on the unsteady Stokes flow of a non-Newtonian fluid in parallel porous plates with periodic injection and suction at the plates under no slip conditions. The governing equations are solved analytically using the stream function, and the velocity components and pressure distribution are examined. The findings demonstrate that non-Newtonian fluids behave very differently from Newtonian fluids in porous channels. The graphical representations of the physical outcomes provide valuable insights into the behavior of non-Newtonian fluids in porous channels. The main finding of this research indicates that the volume flow rate is not affected by the different parameters. This result provides a useful guideline for designing channels with porous walls for the transport of non-Newtonian fluids. The exact solution obtained in this research represents a generalization of previous studies when a non-Newtonian parameter approaches zero.

Keywords: Non-Newtonian Fluid; Unsteady Laminar Flow; Porous Plates; Periodic Injection and Suction; No Slip Conditions.

Introduction

The most significant area of physics is fluid mechanics, which has a significant impact on how we live our daily lives. The vast majority of fluids that engineers and scientists work with, including air, water, and oils, may generally be thought of as Newtonian. However, the considerably more complicated non-Newtonian reaction must be described in many situations where the assumption of Newtonian behaviour is erroneous. These situations can be seen in the industries that

[*]Department of Mathematics and Basic Sciences, Sukkur IBA University, Sukkur 65200, Pakistan, younisalipanhwar@gmail.com

[†]Corresponding Author: Department of Basic Science and Related Studies, Mehran University of Engineering and Technology, SZAB Campus, Khairpur 66020, Pakistan, kaleemullah@muetkhp.edu.pk

[‡]Department of Mathematics and Basic Sciences, Sukkur IBA University, Sukkur 65200, Pakistan, zarqa.bano@iba-suk.edu.pk

[§]Department of Basic Science and Related Studies, The University of Larkano, Larkana 77150, Pakistan, afaq_bhutto@uolrk.edu.pk

produce chemicals and plastics. Applications such as lubricating and biomedical flows, as well as the mining industry, where slurry and mud are regularly handled, also exhibit non-Newtonian behaviour. Therefore, industry places a high value on the Modelling non-Newtonian fluid flow. Unsteady flow has received a lot of attention and impact due to its applicability to many sorts. This occurs during the lubrication process, in lava, paint, viscous polymers, swimming of microorganisms.

Navier (1823) introduced a slip boundary condition where the shear rate at the wall determines the relative velocity of the fluid with the wall. Molecular calculations were used to develop Navier-slip. The Berman (1952) was the pioneer who addressed the problem of steady flow of an in-compressible viscous fluid via a porous rectangular channel in the situation of low Reynolds number. By using the assumption that the normal wall velocities are the same, he found a perturbation solution. Following this, Sellars (1955) went on to further investigate the similar issue for the situation of a high Reynolds number. Gradually, (Yuan, 1956) formalised this problem for a range of injection and suction Reynolds numbers. Terrill (1965) analysed this problem, taking into account various normal velocities at the walls. Drake (1965) investigated the cross-sectional flow of an in-compressible viscous fluid caused by a periodic pressure gradient. Bagchi researched the transient pressure gradient and viscoelastic Maxewell fluid flow in a rectangular channel.

Bhutto et al. (2023) and Khokhar et al. (2023) have investigated that magnetic fields interact with viscous fluid flows under different stream profiles, providing insights into velocity distributions, boundary layer effects, and stability considerations. The mentioned studies are meant to provide further insight into the unsteady stokes flow of Newtonian and non-Newtonian fluid across two parallel porous plates.

The various works (Berman, 1952; Narasimhan, 1961; Sellars, 1955; Donoughe, 1956; Erdogan et al., 2004; Erdogan, 1997) are restricted to a knowledge of the problem related to flow in porous channels under different situations that is simply mathematical. There is further information that supports this article and my research strategy, including two publications on the topics of unsteady in-compressible Stokes flow in porous pipes and porous channel with periodic suction and injection under slip conditions that were both published (Bhatti et al., 2017; Bhatti et al., 2018) . Ganesh (2007) have already addressed this article's problem for the Newtonian case, and we expanded the same work for the non-Newtonian fluid. Suction and injection in tubes play an important role in different fields, including fluid mixing, biological systems, and filtration processes. In fluid mixing, they help achieve uniform blending of liquids or gases in confined spaces, which is important for chemical reactions and industrial processes (Nienow et al., 1997). In biological systems, these techniques are used in microfluidic devices for tasks like delivering nutrients, sampling fluids, or studying cellular behavior with precision (Sivagnanam & Gijs, 2013). In filtration, suction pulls fluids through fine membranes for purification, while injection can introduce cleaning agents to maintain filter performance (Wagner, 2001). These processes offer precise control, making them essential tools in both scientific research and practical applications.

Nomenclature

u	Axial velocity component (m/s)
v	Radial velocity component (m/s)
t	Time (s)
x	Axial coordinate (m)
y	Radial coordinate (m)
μ	Coefficient of viscosity (Pa·s)
α_1, α_2	Non-Newtonian parameters
τ	Stress tensor
A_1, A_2	First and second Rivlin-Erickson tensors
∇^2	Laplacian operator
ω	Frequency (Hz)
L	Length of porous plates (m)
Q	Volume flow rate (m^3/s)
P	Pressure (Pa)
ψ	Stream function
ρ	Density of the fluid (kg/m^3)
R	Reynolds number
κ	Permeability of the porous medium

The parameters used in this work are significant because they influence the behavior of non-Newtonian fluid flow between parallel porous plates under suction and injection directly. Parameters like the non-Newtonian coefficient, viscosity, and frequency determine the axial and radial velocity profiles, highlighting the complex interactions between fluid and porous medium. These parameters allow the study to generalize previous Newtonian fluid models and explore new dynamics in non-Newtonian flows.

Second Grade Fluid Model

A second-grade fluid is a kind of non-Newtonian fluid that is distinguished by a constitutive relation that considers the fluid's earlier deformation in addition to its current rate of strain. Second-grade fluids show a more complex relationship between stress and strain, including higher-order derivatives of the velocity field, in contrast to Newtonian fluids, where the stress is proportionate to the rate of strain.

Mathematically, The stress tensor T can be defined as:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{1}$$

Where μ is the co-efficient of viscosity, α_1 and α_2 are the non-Newtonian parameters. A_1 and A_2 are the first and second Rivlin-Erickson tensors defined as follows:

$$A_1 = \nabla V + (\nabla V)^T, \tag{2}$$

$$A_2 = \frac{\partial A_1}{\partial t} + (V.\nabla)A_1 + (A_1)(\nabla V) + (\nabla V)^T(A_1). \tag{3}$$

Governing Equations

The governing equations for an in-compressible flow of the second-grade fluid by neglecting the thermal effect and body forces are represented by the following:

$$\nabla . V = 0, \tag{4}$$

$$\rho \frac{DV}{Dt} = divT, \tag{5}$$

Where, $\frac{DV}{Dt}$ is material time derivatives, and it is described as:

$$\frac{D(*)}{Dt} = \frac{\partial(*)}{\partial t} + V.\nabla(*), \tag{6}$$

Using (1) in (5) we get the vector form of (5) in the following form:

$$\rho \frac{DV}{Dt} = -\nabla p + \mu \nabla^2 V + \alpha_1 \left[\frac{\partial}{\partial t}\nabla^2 V + \nabla^2(\nabla \times V) + grad\left(V.\nabla^2 V + \frac{1}{4}\left|A_1^2\right|\right)\right] + (\alpha_1 + \alpha_2)div\left|A_1^2\right|, \tag{7}$$

Where, ∇^2 denote the laplacian ,and $|A_1^2| = tr[A_1 A_1^t]$ in the case of unsteady plane coordinates we take,

$$v = \big(u(x,y), v(x,y)\big), \tag{8}$$

Where, u and v represent the axial and radial components of velocity, respectively, putting (8) in (4) and (2) we obtain the following equations as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{9}$$

x-Component

$$\rho \frac{\partial u}{\partial t} - \rho v \Omega = \frac{\partial p^*}{\partial x} + \mu \nabla^2 u + \alpha_1 \frac{\partial}{\partial t}(\nabla^2 u) - \alpha_1 v \nabla^2 \Omega, \tag{10}$$

y-Component

$$\rho \frac{\partial v}{\partial t} + \rho u \Omega = \frac{\partial p^*}{\partial y} + \mu \nabla^2 v + \alpha_1 \frac{\partial}{\partial t}(\nabla^2 v) + \alpha_1 u \nabla^2 \Omega. \tag{11}$$

Where p^* , $|A_1^2|$ and Ω are define as:

$p^* = -p + \frac{\rho}{2}(u^2 + v^2) + \alpha_1(u\nabla^2 u + v\nabla^2 v) + \frac{1}{4}(3\alpha_1 + 2\alpha_2)|A_1^2|,$
$|A_1^2| = tr[A_1 A_1^t] = 8(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2,$
$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$

Problem Description

Consider an in-compressible flow between parallels porous plates at $y = 0$ and $y = L$ in the direction of the x -axis. We assume Stokes flow with periodic injection and suction with velocity $v_1 e^{iwt}$ on the lower and $v_2 e^{iwt}$ on the upper plates respectively, where v_1 and v_2 are constant and w is the frequency which has been shown in the Figure 1.

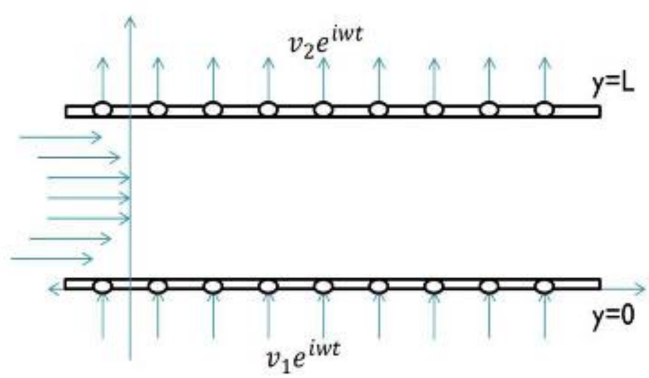

Figure1: Unsteady stokes flow for Non-Newtonian fluid Parallel Porous Plate.

Problem Solution

Consider u and y to be the flow field's axial and radial velocity vectors, respectively, at any time t and choose velocity vector q (Ganesh et al., 2007) which can be written as:

$$\bar{q} = [u(x,y)\hat{\imath} + v(x,y)\hat{\jmath}]e^{iwt},$$
$$u = u(x,y)e^{iwt}, \quad v = v(x,y)e^{iwt},$$
$$P = p(x,y)e^{iwt}. \tag{12}$$

The equation of continuity and equations of motion for non-Newtonian fluid can be reduced to the following by neglect the convective terms from (10) and (11) due to very small Reynolds number. This is mainly due to the assumptions of the Stokes flow regime, where inertial effects are negligible compared to viscous forces.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0, \tag{13}$$
$$\rho\frac{\partial u}{\partial t} = \frac{\partial p^*}{\partial x} + \mu\nabla^2 u + \alpha_1\frac{\partial}{\partial t}(\nabla^2 u) - \alpha_1 v\nabla^2\Omega, \tag{14}$$

$$\rho\frac{\partial v}{\partial t}=\frac{\partial p^*}{\partial y}+\mu\nabla^2 v+\alpha_1\frac{\partial}{\partial t}(\nabla^2 v)+\alpha_1 u\nabla^2\Omega. \quad (15)$$

The boundary condition of the problem are:

$$u(x,0)=0, \quad u(x,L)=0, \quad (16)$$

$$v(x,0)=v_1, \quad v(x,L)=v_2, \quad (17)$$

Where L is length of parallel porous plates. The incompressibility of the fluid, no-slip boundary conditions, and steady periodic injection and suction at the plates suggests introducing the stream function ψ (x,y) as:

$$u(x,y)=\frac{\partial\psi}{\partial y} \quad and \quad v(x,y)=-\frac{\partial\psi}{\partial x}, \quad (18)$$

As continuity equation (13) is satisfied. Equations (14) and (15) can be written as:

$$\rho i\omega u e^{iwt}=e^{iwt}\frac{\partial p^*}{\partial x}+e^{iwt}\mu\nabla^2 u+e^{iwt}\alpha_1 i\omega(\nabla^2 u)-(e^{iwt})^2\alpha_1 v\nabla^2\Omega$$

$$\rho i\omega u=\frac{\partial p^*}{\partial x}+\mu\nabla^2 u+\alpha_1 i\omega(\nabla^2 u)-\alpha_1 e^{iwt}v\nabla^2\Omega, \quad (19)$$

$$\rho i\omega v e^{iwt}=e^{iwt}\frac{\partial p^*}{\partial y}+e^{iwt}\mu\nabla^2 v+e^{iwt}\alpha_1 i\omega(\nabla^2 u)-(e^{iwt})^2\alpha_1 v\nabla^2\Omega$$

$$\rho i\omega v=\frac{\partial p^*}{\partial y}+\mu\nabla^2 v+\alpha_1 i\omega(\nabla^2 v)+\alpha_1 i e^{iwt}u\nabla^2\Omega, \quad (20)$$

and also

$$\Omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

$$\Omega=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2},$$

$$\Omega=-\nabla^2\psi, \quad (21)$$

Using (18) and (21) in equations (19) and (20) we have

$$i\omega\frac{\partial\psi}{\partial y}=\frac{1}{\rho}\frac{\partial p^*}{\partial x}+\nu\nabla^2\left(\frac{\partial\psi}{\partial y}\right)+\frac{\alpha_1}{\rho}i\omega\nabla^2\left(\frac{\partial\psi}{\partial y}\right)-\frac{\alpha_1}{\rho}e^{iwt}\frac{\partial\psi}{\partial x}(\nabla^4\psi), \quad (22)$$

$$-i\omega\frac{\partial\psi}{\partial x}=\frac{1}{\rho}\frac{\partial p^*}{\partial x}-\nu\nabla^2\left(\frac{\partial\psi}{\partial x}\right)-\frac{\alpha_1}{\rho}i\omega\nabla^2\left(\frac{\partial\psi}{\partial x}\right)-\frac{\alpha_1}{\rho}e^{iwt}\frac{\partial\psi}{\partial y}(\nabla^4\psi). \quad (23)$$

Partially differentiating (22) with respect to 'y' we get

$$i\omega\frac{\partial^2\psi}{\partial y^2}=\frac{1}{\rho}\frac{\partial^2 p^*}{\partial x\partial y}+\nu\frac{\partial}{\partial y}\left[\nabla^2\left(\frac{\partial\psi}{\partial y}\right)\right]+\frac{\alpha_1}{\rho}i\omega\frac{\partial}{\partial y}\left[\nabla^2\left(\frac{\partial\psi}{\partial y}\right)\right]-\frac{\alpha_1}{\rho}e^{iwt}\frac{\partial^2\psi}{\partial x\partial y}(\nabla^4\psi),$$

$$\frac{\partial^2 p^*}{\partial x\partial y}=\rho i\omega\frac{\partial^2\psi}{\partial y^2}-\mu\frac{\partial}{\partial y}\left[\nabla^2\left(\frac{\partial\psi}{\partial y}\right)\right]-\alpha_1 i\omega\frac{\partial}{\partial y}\left[\nabla^2\left(\frac{\partial\psi}{\partial y}\right)\right]+\alpha_1 e^{iwt}\frac{\partial^2\psi}{\partial x\partial y}(\nabla^4\psi). \quad (24)$$

Partially differentiating (23) with respect to 'x' we get

$$-i\omega\frac{\partial^2\psi}{\partial x^2}=\frac{1}{\rho}\frac{\partial^2 p^*}{\partial x\partial y}-\nu\frac{\partial}{\partial x}[\nabla^2(\frac{\partial\psi}{\partial x})]-\frac{\alpha_1}{\rho}i\omega\frac{\partial}{\partial x}[\nabla^2(\frac{\partial\psi}{\partial x})-\frac{\alpha_1}{\rho}e^{iwt}\frac{\partial^2\psi}{\partial x\partial y}(\nabla^4\psi)$$

$$2mm\frac{\partial^2 p^*}{\partial x\partial y} = -\rho i\omega\frac{\partial^2\psi}{\partial x^2} + \mu\frac{\partial}{\partial x}[\nabla^2(\frac{\partial\psi}{\partial x})] + \alpha_1 i\omega\frac{\partial}{\partial x}[\nabla^2(\frac{\partial\psi}{\partial x})] + \alpha_1 e^{iwt}\frac{\partial^2\psi}{\partial x\partial y}(\nabla^4\psi). \quad (25)$$

The result obtained by eliminating the pressure p from the equations (24) and (25) is

$$0 = \rho i\omega(\nabla^2\psi) - \left(\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4}\right)(\mu + \alpha_1 i\omega),$$

$$0 = (\nabla^2\psi) - \left(\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4}\right)\left(\frac{\alpha_1 i\omega+\mu}{\rho i\omega}\right). \quad (26)$$

The function f(η) is introduced as:

$$\psi(x,y) = \left(L\frac{u_0}{a} - \nu_2 x\right)f(\eta), \quad (27)$$

Where $\eta = \frac{y}{L}, a = 1 - \frac{v_1}{v_2}, 0 \le v_1 \le v_2$ and u_0 is the average entrance velocity. Equation (27) can be rewritten as:

$$(L\frac{u_0}{a} - \nu_2 x)\frac{f''(\eta)}{L^2} - (L\frac{u_0}{a} - \nu_2 x)\frac{f^{iv}(\eta)}{L^4}(\frac{\mu+\alpha_1 i\omega}{\rho i\omega}) = 0,$$

$$\frac{f''(\eta)}{L^2} - \frac{f^{iv}(\eta)}{L^4}(\frac{\mu+\alpha_1 i\omega}{\rho i\omega}) = 0,$$

$$f^{iv}(\eta) - (\frac{L^2\rho i\omega}{\alpha_1 i\omega+\mu})f''(\eta) = 0,$$

$$f^{iv}(\eta) - \beta^2 f''(\eta) = 0, \quad (28)$$

Where $\beta^2 = (\frac{L^2\rho i\omega}{\alpha_1 i\omega+\mu})$. As equation (28) is forth order ordinary homogeneous linear differential equation and can be solved by converting it into auxiliary equation as:

$$m^2(m^2 - \beta^2) = 0,$$

$$m = 0,0,\beta,-\beta.$$

$$f(\eta) = C_1 + C_2\eta + C_3 e^{\beta\eta} + C_4 e^{-\beta\eta}, \quad (29)$$

The following are the boundary conditions for f(η):

$$f(0) = 1 - a, f(1) = 1 \quad and \quad f'(0) = f'(1) = 0 \quad (30)$$

Finding the values of C_1 to C_4 by using the boundary conditions (30) as:

$$C_1 = 1 - a - \frac{a(e^\beta+e^{-\beta}-2)}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)},\ C_2 = \frac{a\beta(e^\beta-e^{-\beta})}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)},$$

$$C_3 = \frac{a(e^{-\beta}-1)}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)},\ C_4 = \frac{a(e^{\beta}-1)}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)}.$$

Substituting the values of constants in (29)

$$f(\eta) = 1 - a - \frac{a(e^\beta+e^{-\beta}-2)}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)} + \frac{a\beta(e^\beta-e^{-\beta})}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)}\eta + \frac{a(e^{-\beta}-1)}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)}e^{\beta\eta} + \frac{a(e^{\beta}-1)}{4+e^\beta(\beta-2)-e^{-\beta}(\beta+2)}e^{-\beta\eta},$$

Above equation can be written as:

$$f(\eta) = 1 - a - \frac{2a}{2+\beta\sinh(\beta)-2\cosh(\beta)}[cosh(\beta) - \beta\eta\, sinh(\beta)$$

$$+ cosh(\beta\eta) - cosh\,\beta\,(\eta - 1) - 1].$$

Stream function is described as:

$$\psi(x,y) = (L\frac{u_0}{a} - v_2 x) f(\eta),$$

$$u(x,y) = \frac{\partial\psi}{\partial y} \quad and \quad v = -\frac{\partial\psi}{\partial x},$$

$$u = (\frac{u_0}{a} - \frac{v_2 x}{L}) f'(\eta)$$

$$u(x,y) = (\frac{u_0}{a} - \frac{v_2 x}{L})[\frac{-\beta a}{2+\beta\, sinh(\beta) - 2\, cosh(\beta)}](sinh(\beta\eta) - sinh\,\beta\,(\eta - 1) - sinh(\beta))\ ,$$

As $u = u(x,y)e^{i\omega t}$ and $x^* = \frac{x}{L}$ then

$$u = (\frac{u_0}{a} - v_2 x^*)[\frac{-\beta a}{2+\beta\, sinh(\beta) - 2\, cosh(\beta)}](sinh(\beta\eta) - sinh\,\beta\,(\eta - 1) - sinh(\beta))e^{i\omega t}, \quad (31)$$

and also

$$v = -\frac{\partial\psi}{\partial x},\ v = -v_2 f(\eta),$$

then

$$v(x,y) = -v_2(1 - a - \frac{2a}{2+\beta\, sinh(\beta) - 2\, cosh(\beta)})[cosh(\beta) - \beta\eta\, sinh(\beta) + cosh(\beta\eta) - cosh\,\beta\,(\eta - 1) - 1],$$

as $v = v(x,y)e^{i\omega t}$ then

$$v = -v_2(1 - a - \frac{2a}{2+\beta\, sinh(\beta) - 2\, cosh(\beta)})[cosh(\beta) - \beta\eta\, sinh(\beta) + cosh(\beta\eta) - cosh\,\beta\,(\eta - 1) - 1]e^{i\omega t}, \quad (32)$$

For special case we represent the velocity profile (31) and (32) derived by putting $\alpha_1 = 0$ in the value of β and get the same result which achieved by (Ganesh et al., 2007) in his article.

Pressure Distribution

Using the velocity components (31) and (32), the pressure components may be determined as:

$$p^* = -p + \frac{\rho}{2}(u^2 + v^2) + \alpha_1(u\nabla^2 u + v\nabla^2 v) + \frac{1}{4}(3\alpha_1 + 2\alpha_2)|A_1^2|,$$

Where $|A_1^2| = tr[A_1 A_1^t] = 8(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2$.

Results and Discussion

The values of the axial and radial velocities were computed for different values of wt, The findings are displayed in Figures 2, 3, and 4 are the axial and Figures 4 and 5 are radial velocity profiles at different values of the parameters at $x^* = 2$, $\beta = 2,4,6$, $a = 2$, $v_2 = 1,2$ and the average entrance velocity is taken to be $u_0 = 0.5$. It is seen clearly from the figures 2, 3 and 3 when we increase the value of β from 2 to 6 then the velocity profiles decrease but the behavior of the velocity remain same.

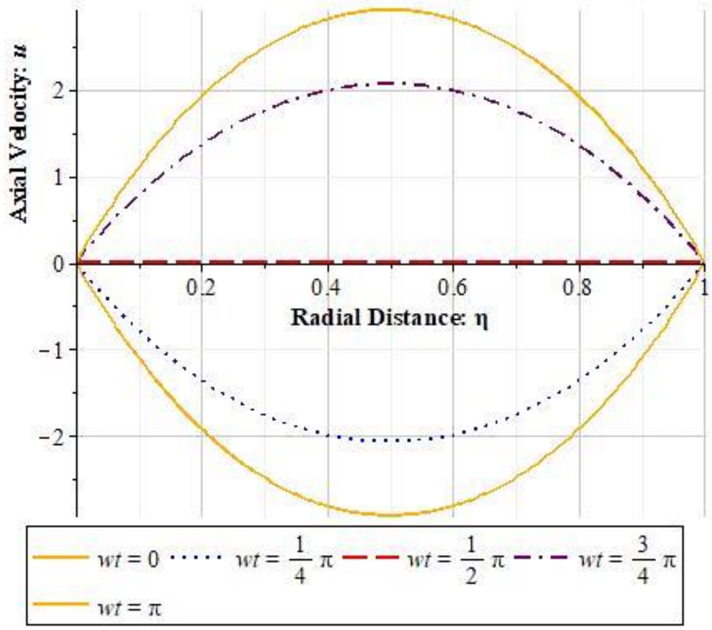

Figure 2: Axial velocity for different values of "wt" when $\boldsymbol{\beta = 2}$.

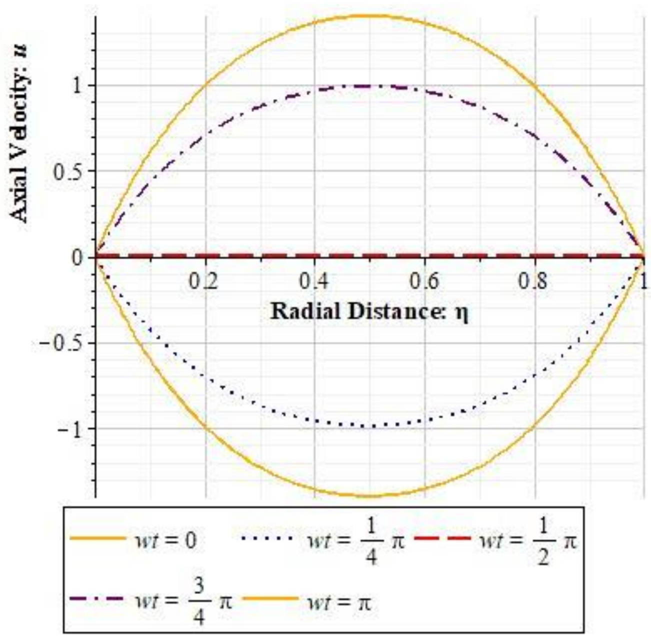

Figure 3: Axial velocity for different values of "wt" when $\boldsymbol{\beta = 4}$.

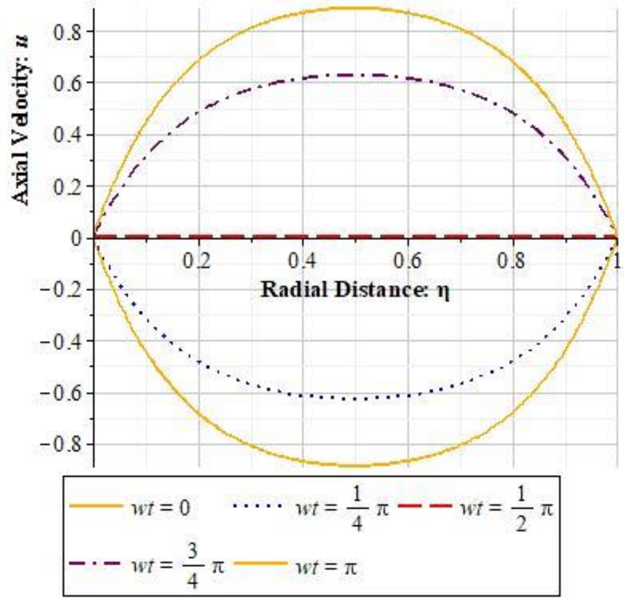

Figure 4: Axial velocity for different values of "wt" when $\beta = 6$.

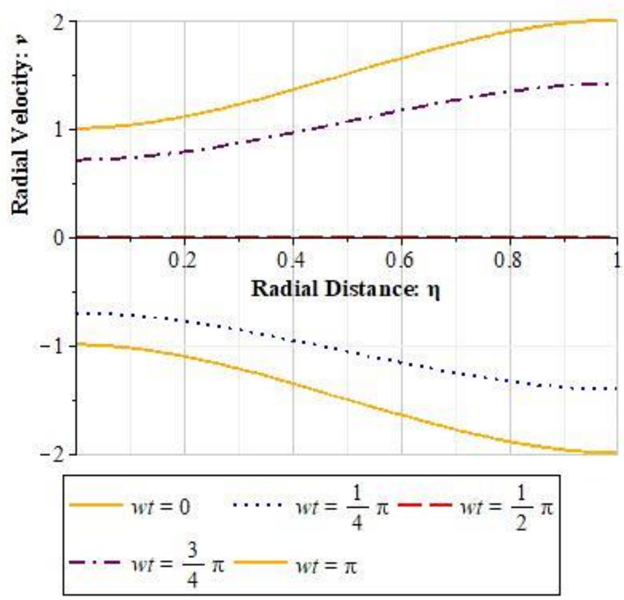

Figure 5: Radial velocity for different values of "wt" when $\beta = 2$.

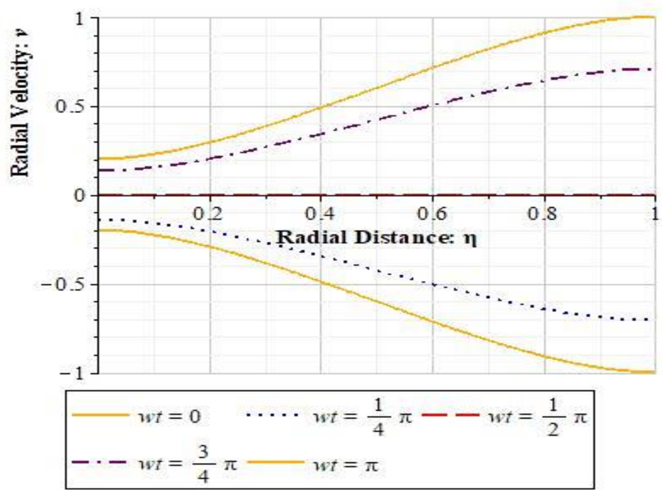

Figure 6: Radial velocity for different values of "wt" when $\boldsymbol{\beta = 4}$.

It has also been noted that the magnitude of the axial velocity increase at $wt = \frac{3}{4}\pi$ and $wt = \pi$. The velocity has reverse flow at the $wt = \frac{1}{4}\pi$ and $wt = 0$. It is seen from figures 5 and 6 that if we increase the value of $\beta = 2,4$ which include non-Newtonian parameter α_1 then the radial velocity increased. It is noted that axial and radial velocities vanish at $wt = \frac{\pi}{2}$ and non-zero on different value of wt.

The overall findings of this study provide understanding the behavior of unsteady flow of non-Newtonian fluids across porous media. The graphical representations of the velocity profiles provide a clear visualization of the effects of different physical parameters, including β, v_2, and wt, on the fluid flow. These results can be useful for designing and optimizing fluid flow systems in various industrial and environmental applications.

Conclusion and Future Work

In conclusion, this study has provided an exact solution for the velocity fields of a non-Newtonian fluid flowing via a porous channel with a uniform cross section under no slip conditions. The results of this study shed light on the impact of the parameter β, which includes the non-Newtonian parameter α_1, on the velocity profiles. Firstly, it was found that when the value of β increases while the flow rate remains constant, the axial velocity on the center-line of channel decreases, while the velocity of the fluid layers in contact with the channel walls increases. The explanation for this phenomena is that increasing the value of β results in a decrease in the size of the pores between the porous medium, which leads to an increase in frictional forces between the fluid and the channel walls. Secondly, it was observed that when the non-Newtonian parameter α_1 is set to zero in the β parameter, the results obtained are consistent with the findings of a previous study by Ghanesh. This observation highlights the importance of the non-Newtonian parameter in the flow of non-Newtonian fluids through porous media. Thirdly, the study found that both the axial and radial velocities vanish at $wt = \pi/2$ and are non-zero at different values of wt. This result suggests that the velocity profiles of non-Newtonian fluids through porous channels exhibit complex behavior, which should be taken into account when designing and optimizing porous media systems.

Furthermore, it was noticed that as the values of the various parameters increase, the size of the velocities inside the channel decreases. This observation indicates that the flow of non-Newtonian fluids through porous media is highly sensitive to changes in the system parameters, and therefore, the optimization of these parameters is crucial to achieve optimal performance. Finally, the study demonstrated that non-Newtonian parameters and other factors have no effect on the volume flow rate. This result implies that the volume flow rate through porous media can be predicted solely based on the physical properties of the fluid and the geometry of the channel, which is an important finding for the design and optimization of porous media systems.

This study provides valuable physical insights into the flow of non-Newtonian fluids through porous media, which can inform the design and optimization of a wide range of engineering systems, including filtration, separation, and heat transfer applications.

Future research in this work may be adding thermal effects, variable porosity, and non-uniform suction and injection to add the understanding the flow of the complex non-Newtonian fluid. Extending the study to 3D and more complex flows like turbulent flows or experimentally validating of the model could be the future advancements. These would benefit the fields like energy, filtration, and biomedical engineering.

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