An Enhanced Solution of Logistic Model by Applying Differential Transform Method

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Abstract
The Differential Transform Method (DTM) is a numerical technique, which results in an analytical solution in the form of polynomials. Although, the method is applied to number of linear and nonlinear boundary and initial value problems, but its generalization needs to be further investigated. This technique is capable of reducing the size and time of computational work in obtaining the exact solutions. The method is used for finding the coefficients for the Taylor's series. The efficiency and easiness of the method is investigated over the logistic problems. The approximated results are compared with that of exact solutions. The numerical results obtained through this technique in this research work are found in good agreement to the existing exact solutions which is very clearly presented in the form of tables and graphs. It has been perceived accordingly that the DTM is easier and effective technique to get a series solution in the form of variables, and the results have shown incredible performance of the proposed method. The numerical results and its comparison also show that the proposed method gives realistic results which is more effective for solving logistic problems.

***Keywords**:* Differential Transform Method, Initial Value Problems, Exact solutions, Logistic Problems.

Introduction

Differential equations have been focused in many areas of scientific studies, due to their consistent applications in physics, fluid mechanics , visco-elastic fluid (Khan et al., 2022), biology, nanofluid (Khan et al., 2024) combustion theory, chemistry, and chemical engineering. Theses equations arising in the theory of mass and heat transfer (Jan, Ullah, Alam, et al., 2024; Jan, Ullah, Khan, et al., 2024; M. M. Naeem et al., 2016; Ullah et al., 2021, 2022), nonlinear mechanics, theory of nonlinear oscillations, elasticity, hydrodynamics, combustion theory, and chemical engineering (Carr, 2006). Almost in all these areas,

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linear and nonlinear differential equations arises (Polyanin & Zaitsev, 2017). Most of the differential equations are nonlinear that's why it is not very easy to solve them analytically. A solution of differential equation is that set of functions which satisfies the differential equation. The simplest differential equations are mostly solvable by explicit formulas; if a formula for the solution is not available then the numerical solution may be approximated using Computers Software. The analytical solution of differential equation cannot be achieved all the times. The equations arising in engineering and other sciences are solved analytically as well as by numerical methods, but the numerical methods can solve both nonlinear and linear problems as well. The solutions of differential equations with constant coefficients, is easy to attain the approximated solutions but it is not simple to find the closed form solutions for high ordered differential equations. In this situation the numerical methods and power series are used to solve the differential equations. Dawar and Hamid (Dawar et al., 2023; Dawar & Khan, 2025) successfully applied Improved Residual Power Series for the series solution, while Naeem et. al (M. Naeem et al., 2016) used for boundary value problems.

Logistic equations have been introduced by a Belgian mathematician Verhulst in 1838, which is a type of generalization of the equation of exponential growth rate (Bacaër, 2011). The logistic equation has been successfully applied to describe the growth of populations in both the laboratory natural habitat, limiting the growth by influence of competition, fertility factors and mortality (Camargo et al., 2014). The distinctive uses of logistic graphs are in the area of medical science, where the logistic differential equation is handled to model the growth rate of Tumors (Vivek et al., 2016). The logistics differential equation is applied in electrochemical dynamics (Sadkowski, 2000).

A new logistic model was recently developed in the study which is found to be useful for bacterial growth predictions at constant and dynamic temperatures, and successfully labelled growth curves of bacteria under different conditions (Fujikawa et al., 2004). In this research study the basic idea of Differential Transform Method (DTM) will be analyzed and applied on different initial values problems. DTM will be used to find the approximated solutions of logistic equations, linear, nonlinear and system of differential equations. DTM is simple in principles and efficient.

Chinese mathematician Zhou introduced DTM for the first time in 1986 (Siddiqi et al., 2012). It was used for the solution of the nonlinear and linear initial value problems aroused in electrical circuit study. This is a semi analytical numerical method used Taylor series for finding the solution of differential equation in the form of polynomials. DTM is an additional way for obtaining Taylor series solution of a differential equation. The series obtained by DTM is recurrently coincides with Taylor expansion of the true solution at $t = 0$ in initial value problems. But it is dissimilar from traditional high ordered Taylor's series method, that requires symbolic calculation of the desired derivatives of the data functions (Applications, 2010).

Two dimensional DTM has been used for the numerical solutions of the initial value problems and some new theorems have been added ("On the Two-Dimensional DTM," 2003). Some linear and nonlinear PDEs, were solved by the method (Rawashdeh, 2013) . The method is applied to linear and nonlinear problems which reduced the size of the computational work. The result obtained by the method is compared with Decomposition Method. It reduces the size of computational work and can easily be applied to many problems. This method was used for the solution of partial differential equations (Benhammouda, 2023). The same author applied DTM for the algebraic equation of index-1. Liu and song critically analyzed that the method is suitable for algebraic differential equation of index-2 but not suitable for the solutions of DAEs of index-3 (Lu et al., 2008).

Analysis of the Differential Transform Method

To give a clear overview of the basic concept of the DTM method, let us consider a function$y(t)$ its differential transform is given as

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0}, \tag{1}$$

$Y(k)$ is the transformed function of the original function $y(t)$ which is also called T-function. Usually, the original function is denoted by small letter and the Transform functions by capital letters

Inverse Transform of this function can be defined as

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k, \tag{2}$$

Combining these two equations

$$Y(k) = \sum_{k=0}^{\infty} \frac{d^k y(t)}{dt^k}\frac{1}{k!}t^k. \tag{3}$$

The fundamental theorems of DTM

1) Let $z(t) = x_1(t) \mp x_2(t)$ then its differential transform is $Z(k) = X_1(k) \mp X_2(k)$.

Proof:

By definition (1), we have

$$X_1(k) = \frac{1}{k!}\left[\frac{d^k x_1(t)}{dt^k}\right]_{t=0}$$

$$X_2(k) = \frac{1}{k!}\left[\frac{d^k x_2(t)}{dt^k}\right]_{t=0}$$

$$Z(k) = \frac{1}{k!}\left[\frac{d^k x_1(t)}{dt^k} \mp \frac{d^k x_2(t)}{dt^k}\right]_{t=0}$$

$Z(k) = X_1(k) \mp X_2(k)$ is proved.

2) For $g(t) = \ell y(t)$, then $G(k) = \ell Y(k)$ where λ is constant

Proof: We have the defination (1)

$$\Rightarrow Z(k) = \frac{1}{k!}\left[\frac{d^k \ell y(t)}{dt^k}\right]_{t=0}$$

$$\Rightarrow Z(k) = \ell \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0}$$

$\Rightarrow Z(k) = \ell Y(k)$ it completes the proof.

3) For $h(t) = \frac{d}{dt} f(t)$, then $H(k) = (k+1)F(k+1)$

4) For $z(x) = \frac{d^m}{dx^m} y(x)$, then $Z(k) = \frac{(k+m)!}{k!} Y(k+m)$

Using definition (1)

$$Z(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\left(\frac{d^m}{dt^m} y(t)\right)\right]_{t=0}$$

$$Z(k) = \frac{1}{k!}\left[\frac{d^{k+m}}{dt^{k+m}} y(t)\right]$$

$$Z(k) = \frac{(k+1)(k+2)\ldots\ldots\ldots(k+m)}{k!(k+1)(k+2)\ldots\ldots\ldots(k+m)}\left[\frac{d^{k+m}}{dt^{k+m}} y(t)\right]$$

$$Z(k) = \frac{(k+1)(k+2)\ldots\ldots\ldots(k+m)}{(k+m)!}\left[\frac{d^{k+m}}{dt^{k+m}} y(t)\right]$$

$$Z(k) = \frac{(k+1)(k+2)\ldots\ldots\ldots(k+n)}{(k+m)!} Y(k+n)$$

$$Z(k) = \frac{(k+m)!}{k!} Y(k+m)$$

5) For $f(t) = h_1(t)h_2(t)$, then its DTM is $F(k) = \sum_{\gamma=0}^{k} H_1(\gamma)H_2(k-\gamma)$

$$F(0) = \frac{1}{0!}[h_1(t)h_2(t)]_{t=0} = H_1(0)H_1(0)$$

$$F(1) = \frac{1}{1!}\left[\frac{d}{dt} y(t)w(t)\right]_{t=0}$$

$$F(1) = \left[\frac{dh_1(t)}{dt} h_2(t) + h_1(t)\frac{d}{dt} h_2(t)\right]_{t=0}$$

$$F(1) = H_1(1)H_2(0) + H_1(0)H_2(1)$$

$$F(2) = \frac{1}{2!}\left[\frac{d^2}{dt^2} h_1(t)h_2(t)\right]_{t=0}$$

$$F(2) = H_1(2)H_2(0) + H_1(1)H_2(1) + H_1(0)H_2(2)$$

Generalizing

$$F(k) = \sum_{\gamma=0}^{k} H_1(\gamma)H_2(k-\gamma)$$

6) For $z(x) = x^n$, then $Z(k) = \begin{cases} 0 if n \neq k \\ 1 if n = k \end{cases}$

7) For $z(x) = e^{mx}$, then $Z(k) = \frac{m^k}{k!}$

8) $z(x) = (1+x)^p$, then $Z(k) = \frac{p(p-1)\ldots\ldots\ldots(p-k+1)}{k!}$

9) For $z(t) = Sin(\mu t + \psi)$, then $Z(k) = \frac{\mu^k}{k!} Sin(\frac{\pi k}{2} + \psi)$
Using definition
For $Z(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k} Sin(\mu t + \psi)\right]_{t=0}$
then $Z(k) = \frac{\mu^k}{k!} Sin(\frac{\pi k}{2} + \psi)$
10) For $z(t) = Cos(\mu t + \psi)$, then $Z(k) = \frac{\mu^k}{k!} Cos(\frac{\pi}{2} k + \psi)$
Using deinition
For $Z(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k} Cos(\mu t + \psi)\right]_{t=0}$
then $Z(k) = \frac{\mu^k}{k!} Cos(\frac{\pi}{2} k + \psi)$

Logistic Model

The following model was first proposed by Pierre Verhulst in 1938 (Bacaër, 2011). It describes the population growth

$$\frac{dN}{dt} = \gamma N(1 - \frac{N}{\kappa})$$

N (t) symbolizes population at time “t”, whereas $\gamma > 0$is called Malthusian parameter showing growth rate. For $y = \frac{N}{\kappa}$gives differential equation listed below

$$\frac{dy}{dt} = \gamma y(1 - y)$$

which is known as logistic equation.

Numerical solution of Logistic Equation

For this research work DTM is applied to construct the approximated solution for the Logistic Model (Bhalekar & Daftardar-Gejji, 2012) of the form

$$\left.\begin{array}{l} D^{\beta} y(t) = \gamma y(t)(1 - y(t)). \beta = 1, t > 0, \gamma > 0, \\ where the initial condition \\ y(0) = t_o \end{array}\right\} \quad (4)$$

The exact solution of the given model for $\beta = 1$is

$$y(t) = \frac{e^{\gamma t}}{1 + e^{\gamma t}}$$

Different cases are solved to get affective results.

Case 1

Let us consider the first case where $\gamma = y(0) = \frac{1}{2}$

$D^{\beta} y(t) = \frac{1}{2} y(t)\big(1 - y(t)\big)$, then $\beta = 1, t > 0, \gamma > 0,$

With initial conditions

$$y(0) = \frac{1}{2}. \tag{5}$$

Eq. 5 is transformed by using DTM

$$\left.\begin{aligned} &\frac{(k+1)!}{k!}Y(k+1) = \frac{1}{2}\left(-\sum_{r=0}^{k} Y(r)Y(k-r) + Y(k)\right),\\ &Y(k+1) = \frac{1}{k+1}\left(\frac{1}{2}\left(-\sum_{r=0}^{k} Y(r)Y(k-r) + Y(k)\right)\right),\\ &Y(k+1) = \frac{1}{k+1}\left(\frac{1}{2}\left(-\sum_{r=0}^{k} Y(r)Y(k-r) + Y(k)\right)\right),\end{aligned}\right\} \tag{6}$$

and $Y(0) = \frac{1}{2}$

From Eq. 6, an approximate series solution is obtained

$$\mathrm{y(t)} = \frac{1}{2} + \frac{t}{8} - \frac{t^3}{384} + \frac{t^5}{15360} - \frac{17t^7}{10321920} + \frac{31t^9}{743178240} \tag{7}$$

Case 2

$$D^{\beta}y(t) = \frac{1}{3}y(t)\big(1 - y(t)\big), \beta = 1, t > 0, \gamma > 0, \tag{8}$$

$$y(0) = \frac{1}{2}.$$

Eq. 8 is transformed by applying DTM theorems

For $\beta = 1, y(0) = \frac{1}{2}$ and $\gamma = \frac{1}{3}$,

$$\frac{(k+1)!}{k!}Y(k+1) = \frac{1}{3}\left(-\sum_{r=0}^{k} Y(r)Y(k-r) + Y(k)\right),$$

$$\Rightarrow Y(k+1) = \frac{1}{k+1}\left(\frac{1}{3}\left(-\sum_{r=0}^{k} Y(r)Y(k-r) + Y(k)\right)\right), \tag{9}$$

$$Y(0) = \frac{1}{2}.$$

From Eq. 9, an approximated series solution is established

$$\mathrm{y(t)} = \frac{1}{2} + \frac{t}{12} - \frac{t^3}{1296} + \frac{t^5}{116640} - \frac{17t^7}{176359680} + \frac{31t^9}{28570268160} \tag{10}$$

Case 3

$$\left.\begin{aligned} &D^{\beta}y(t) = \gamma y(t)\big(1 - y(t)\big), \quad \beta = 1 and t > 0,\\ &y(0) = \frac{1}{2}.\end{aligned}\right\} \tag{11}$$

Eq. 11 is transformed by using the above theorems as follows,

For $\beta = 1, Y(0) = \frac{1}{2}$ and $\gamma = \frac{1}{4}$

$$\frac{(k+1)!}{k!}Y(k+1) = \frac{1}{4}\left(Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r)\right),$$

$$\Rightarrow Y(k+1) = \frac{1}{k+1}\left(\frac{1}{4}\left(Y(k) - \sum_{\gamma=0}^{k} Y(r)Y(k-r)\right)\right), \tag{12}$$

and $(0) = \frac{1}{2}$.

From Eq. 12, an approximated series solution is found

$$\mathrm{y(t)} = \frac{1}{2} + \frac{t}{16} - \frac{t^3}{3072} + \frac{t^5}{491520} - \frac{17t^7}{1321205760} + \frac{31t^9}{380507258880}. \tag{13}$$

.

Case 4

$$D^{\beta}y(t) = \gamma y(t)(1 - y(t))\beta = 1, \text{ where } t > 0, y(0) = \frac{1}{2}. \quad (14)$$

Eq 14 is transformed by using the above DTM theorems as follows

For $\beta = 1, y(0) = \frac{1}{2}$ and $\gamma = \frac{1}{8}$

$$\frac{(k+1)!}{k!}Y(k+1) = \frac{1}{8}\left(-\sum_{r=0}^{k} Y(r)Y(k-r) + Y(k)\right),$$

$$\Rightarrow Y(k+1) = \frac{1}{k+1}\left(\frac{1}{8}\left(Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r)\right)\right), \quad (15)$$

$$\text{and } Y(0) = \frac{1}{2}$$

From equations 15, an approximated series solution of order 10 of the initial value problem 14 is given by

$$\mathrm{y(t)} = \frac{1}{2} + \frac{t}{32} - \frac{t^3}{24576} + \frac{t^5}{15728640} - \frac{17t^7}{169114337280} + \frac{31t^9}{194819716546560} \quad (16)$$

The numerical results so obtain are compared with that of exact solution in Table 1 given below. It shows the exact solution, DTM solution and residual which is the difference between both these results.

Table 1: Exact and DTM solution of case 1.

T	Exact solution	DTM Solution	Error
0	0.5	0.5	0.
0.01	0.50125	0.50125	1.97495×10^{-17}
0.02	0.5025	0.5025	-5.42532×10^{-17}
0.03	0.50375	0.50375	-5.18877×10^{-17}
0.04	0.505	0.505	2.92554×10^{-17}
0.05	0.50625	0.50625	9.48156×10^{-17}
0.06	0.507499	0.507499	4.58652×10^{-17}
0.07	0.508749	0.508749	9.73649×10^{-17}
0.08	0.509999	0.509999	6.8902×10^{-17}
0.09	0.511248	0.511248	-6.33802×10^{-17}
0.1	0.512497	0.512497	-1.2207×10^{-17}

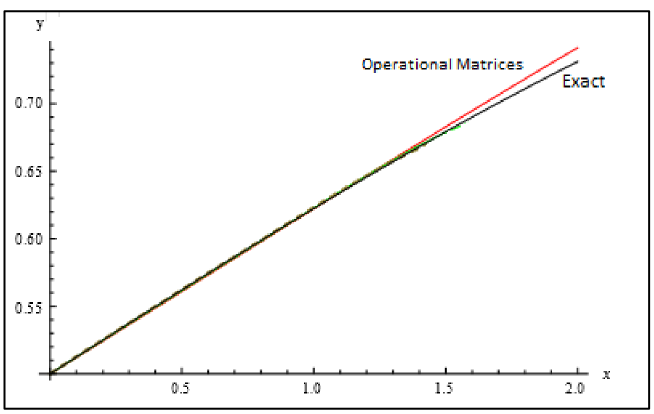

Figure 1: Graph of the Operational Matrices and Exact solution.

Table 1 shows the comparative analysis of Exact and the DTM Solutions. The error illustrates the difference of the DTM and exact solutions. The problem case 1. was solved by Operational Matrices of Bernstein polynomials, Figure 1 is given above. Which shows the rate of convergence. The Figure 2 shows better approximation comparing with Figure 1 and illustrates that the convergence of the DTM approximations, exact solution and specifying the point where divergence starts.

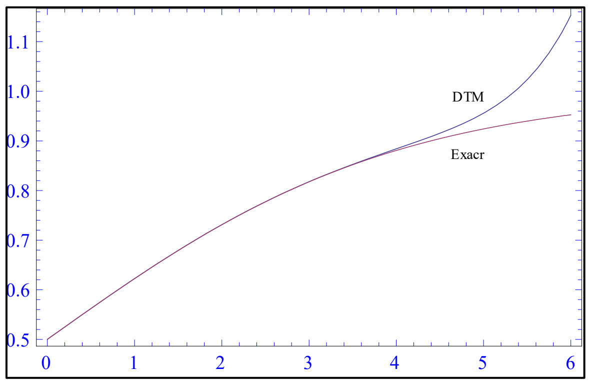

Figure 2: DTM and Exact Solutions.

Table 2 given below displays the detailed comparison of the exact, DTM solution and the difference between the exact, and the approximated DTM solutions studied in case 2. The Figure 3 demonstrates the convergence of the approximated DTM and exact solution, which specifies the point where divergence starts. Numerical results along with error of the problem case 4 is presented in Table 4.

Table 2: Exact and DTM solutions of case 2.

T	Exact solution	DTM Solution	Error
0	0.5	0.5	0.
0.01	0.50125	0.50125	1.97495×10-17
0.02	0.5025	0.5025	-5.42532×10-17
0.03	0.50375	0.50375	-5.18877×10-17
0.04	0.505	0.505	2.92554×10-17
0.05	0.50625	0.50625	9.48156×10-17
0.06	0.507499	0.507499	4.58652×10-17
0.07	0.508749	0.508749	9.73649×10-17
0.08	0.509999	0.509999	6.8902×10-17
0.09	0.511248	0.511248	-6.33802×10-17
0.1	0.512497	0.512497	-1.2207×10-17

Figure 5 shows the convergence and point of divergence of the exact and DTM solutions. From all these results presented in the form of tables and graphs it can easily be concluded that as γ decreases the error decreases and the rate of convergence increases.

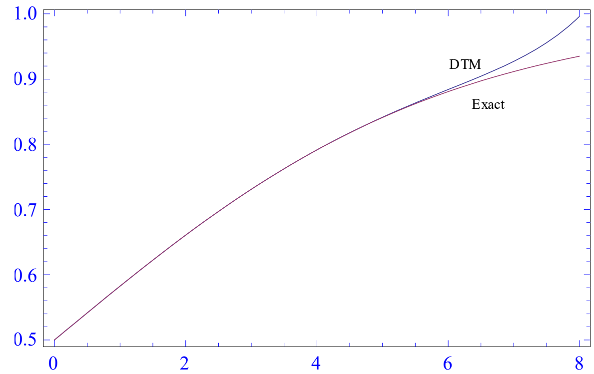

Figure 3: Exact and DTM comparison of problem discussed in case 2.

Table 3 Exact and DTM solution.

T	Exact solution	Solution by DTM	Error
0	0.5	0.5	0.
0.01	0.50625	0.50625	9.48156×10-17
0.02	0.512497	0.512497	-1.2207×10-17
0.03	0.518741	0.518741	3.59758×10-17
0.04	0.524979	0.524979	4.63896×10-18
0.05	0.531209	0.531209	-2.19204×10-16
0.06	0.53743	0.53743	-1.92931×10-15
0.07	0.543639	0.543639	1.01712×10-14
0.08	0.549834	0.549834	-4.4095×10-14
0.09	0.556014	0.556014	-1.61096×10-13
0.1	0.562177	0.562177	-5.12694×10-13

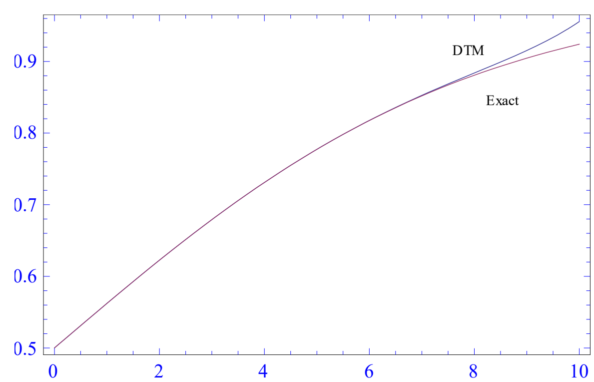

Figure 4: Comparative analysis of the logistic problem case 3.

Figure 4 illustrates the graphical comparison between the exact and DTM solutions of the problem. The plot not only demonstrates the close agreement of both solutions in the initial domain but also clearly highlights the stage at which their behaviors begin to diverge, thereby indicating the range of validity for the DTM approach."

Table 4 Exact and DTM solutions.

T	Exact solution	Solution by DTM	Error
0	0.5	0.5	0.
0.01	0.503125	0.503125	-2.3126×10-17
0.02	0.50625	0.50625	9.48156×10-17
0.03	0.509374	0.509374	-2.67737×10-17
0.04	0.512497	0.512497	-1.2207×10-17
0.05	0.51562	0.51562	-5.18587×10-17
0.06	0.518741	0.518741	3.59758×10-17
0.07	0.521861	0.521861	1.80788×10-17
0.08	0.524979	0.524979	4.63896×10-18
0.09	0.528095	0.528095	-1.69025×10-16
0.1	0.531209	0.531209	-2.19204×10-16

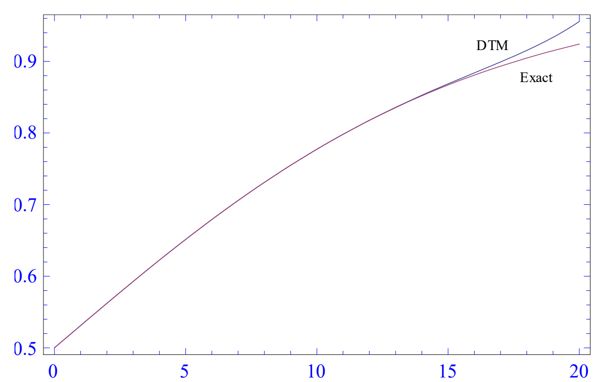

Figure 5: DTM and Exact solution of the logistic problem case 4.

Conclusion

In this paper, the DTM is applied successfully to get the numerical solution of logistic initial value problem. The tabular and graphical comparison between the approximated solutions reveals that DTM is an excellent technique for the solutions of logistic initial value problems. The numerical results and its comparison also show that the proposed method gives authentic results which are more effective, powerful and simple. It is advisable to use it for the investigation of logistics model, DTM delivers more accurate series solutions.

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