Differential Transform Method for Linear and Nonlinear Differential Equations: Convergence, Error Analysis, and Comparative Validation

Hamid Khan[*], Refat Ullah Jan[†], Nudrat Aamir[‡], Subhan Ullah[§], Farah Jaffar[**]

Abstract

The Differential Transform Method (DTM) is a semi-analytical series technique for constructing approximated series solutions of ordinary differential equations without evaluating high-order derivatives as in the Taylor procedure. This study presents a combined assessment of DTM for linear and nonlinear initial-value problems, with particular emphasis on convergence, truncation, and quantitative error evaluation. In addition to comparisons with exact solutions, the revised analysis introduces an analytic truncation-error bound for solutions that are analytic near the expansion point, an explicit stopping criterion, relative and maximum absolute error measures, and an order-normalized comparison with HAM, HPM, and OHAM for a common test problem. A nonlinear Riccati problem with a finite radius of convergence is also included to examine strength and to illustrate limitation of local power-series approximations. The results show that DTM can produce highly precise approximations within its region of convergence, but its performance depends on truncation order, solution analyticity, and the distance from singularities. The study therefore, emphasizes reproducible accuracy assessment rather than an unconditional claim of superiority over other semi-analytical methods.

Keywords: Differential Transform Method, Linear and Nonlinear Differential Equations, Convergence, Truncation Error, Error Analysis, Semi-Analytical Methods

Introduction

Differential equations play an important role in modeling, the fluid mechanics, heat and mass transfer, elasticity, chemical engineering, oscillation theory, and many other scientific and engineering problems (Carr, 2006; Polyanin & Zaitsev, 2017). Closed-form solutions are available only for constrained classes of problems, so approximated

[*]Department of Mathematics, Islamia College Peshawar, Peshawar 25120, Pakistan, hamid@icp.edu.pk

[†]Corresponding Author: CECOS University of IT & Emerging Sciences, Peshawar 25000, Pakistan, refatullah@cecos.edu.pk

[‡]Sarhad University of Science & Information Technology, Islamabad 44000, Pakistan, cd.isb@suit.edu.pk

[§]Department of Mathematics, University of Malakand, Chakdara 23050, Pakistan, subhan.bcst@gmail.com

[**]CECOS University of IT & Emerging Sciences, Peshawar 25000, Pakistan, farah.jaffer@cecos.edu.pk

analytical and numerical techniques remain important for linear and nonlinear models. Common techniques like Runge-Kutta schemes, the Adomian decomposition method, the variational iteration method, the homotopy perturbation method, and the homotopy analysis family (Hashim et al., 2006; Nawaz, 2011).

The Differential Transform is a power-series method related to the Taylor expansion. It generates the series coefficients through algebraic recurrence relations obtained directly from the governing differential equation. In this sense, Differential Transform Method (DTM) avoids the repeated symbolic differentiation required by a conventional high-order Taylor series and can be suitable for initial-value problems. The method has been applied for ordinary differential equations, linear nonlinear partial differential equations, higher-order boundary-value problems, differential-algebraic systems, and fractional models (Odibat & Momani, 2008; Rawashdeh, 2013; Warade & Rastogi, 2018).

Previous studies have confirmed several extensions and applications of DTM. Two-dimensional forms have been developed for partial differential equations, generalized forms have been utilized for the solutions of fractional-order problems, and modified variants have been applied to nonlinear systems (Odibat & Momani, 2008; Benhammouda et al., 2014). DTM has also been applied to compared with Taylor-series and other semi-analytical methods for ordinary differential equations (Yazdani et al., 2016). These studies found the method as a valuable coefficient-generator for power series, but the stated demonstrations often use variuos truncation orders, error definitions, domains and comparison conditions.

The specific contribution of this study is not the development of DTM itself. Rather, the novelty is a unified validation framework for representative linear and nonlinear test problems. This research study state (i) an analytic convergence and truncation-error estimate for analytical solutions, (ii) it specifies the truncation order and a practical stopping criterion, (iii) it also reports relative, absolute and maximum absolute errors, (iv) and introduces an order-normalized comparison with HPM, HAM, and OHAM, and (v) adds a nonlinear Riccati test problem whose exact solution has a finite singularity. This latter test makes the necessity of DTM on the radius of convergence explicit and provides more steady assessment of both strengths and limitations.

Accordingly, the aim is to evaluate when DTM provides reliable local series approximations and how its accuracy changes with truncation and problem structure. Comparisons with other methods are interpreted under stated approximation conditions, and claims of universal superiority in accuracy, efficiency, or stability are avoided unless the computational conditions are directly comparable.

Basic Theorems of Differential Transform Method (DTM)

To provide a concise outline of the fundamental idea of DTM. Let us consider a function $y(t)$, its differential transform about $t = 0$, is defined as

$$Y_1(k) = \frac{1}{k!}\left[\frac{d^k y_1(t)}{dt^k}\right]_{t=0}. \quad (1)$$

Here, $Y_1(k)$ is the T-function or transformed, of the function $y_1(t)$. Generally, the original function is symbolized by capital letter and the Transform functions by small letters. Its Inverse Transform is defined as:

$$y_1(t) = \sum_{k=0}^{\infty} Y_1(k)t^k \quad (2)$$

Combining Equation (1) and (2):

$$Y_1(k) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k y_1(t)}{dt^k}\left(t^k\right) \quad (3)$$

Basic Results of Differential Transform Method (DTM)

1) Let $z_1(t) = y_1(t) \pm y_2(t)$, its transform is $Z_1(k) = Y_1(k) \mp Y_2(k)$.
2) For $g_A(t) = \eta y_A(t)$, then $G_A(k) = \eta Y_A(k)$, where is ηconstant.
3) For $h_A(t) = \frac{d}{dt}f_A(t)$, then $H_A(k) = (k+1)F_A(k+1)$.
4) For $h_A(x) = \frac{d^m}{dx^m}y_A(x)$, then its T-funcion is given below $H_A(k) = \frac{(k+m)!}{k!}Y_A(k+m)$.
5) For $f(t) = g_1(t)g_2(t)$, then its transform will be $F(k) = \sum_{\gamma=0}^{k} G_1(\gamma)G_2(k-\gamma)$.
6) For $f(x) = x^n$, then $F(k) = \begin{cases} 1 if n = k \\ 0 if n \neq k \end{cases}$.
7) If $z(x) = e^{\mu x}$, then its transform $Z(k) = \frac{\mu^k}{k!}$.
8) If $z(x) = (1+x)^q$, then its differential transform will take the form $Z(k) = \frac{q(q-1)\ldots\ldots\ldots(q-k+1)}{k!}$.
9) The transform form of will be
10) The transform of the cosine function $z(t) = Cos(\lambda t + \rho_1)$, will be $Z(k) = \frac{\lambda^k}{k!}Cos(\frac{\pi}{2}k + \rho_1)$.

Convergence, Truncation, and Error Measures

The DTM coefficients are the coefficients of Taylor of the solution at the expansion point; convergence is governed by analyticity. Let $y(t) = \sum_{k=0}^{\infty} Y(k)(t-t_0)^k$be analytic in $|t-t_0| < R$ and assume $|y(z)| \leq M$on the circle $|z-t_0| = R$. Cauchy's estimate gives$|Y(k)| \leq \frac{M}{R^k}$. Hence, for $r = |t-t_0| < R$, the truncation remainder after order N

satisfies $|R_N(t)| \leq \frac{M\left(\frac{r}{R}\right)^{N+1}}{1-\frac{r}{R}}$. This estimation explains, that how the DTM series converges faster, well inside the nearest singularity but can converge slowly as r approaches R. For the numerical calculations, the truncated approximation is $y_N(t) = \sum_{k=0}^{N} Y(k)(t-t_0)^k$. The approximation order is increased until the largest contribution of the newly added term on the reported interval satisfies $max_t|Y(N)(t-t_0)^N| < \varepsilon$, or until a stated fixed order is reached for an order-matched comparison. In the revised calculations, $\varepsilon = 10^{-8}$ is used where the criterion is attainable on the selected interval. Problem 2 uses $N = 10$, for the high-accuracy DTM table, while the cross-method comparison uses the common polynomial degree $N = 3$, for all four methods. Three error measures are used. The absolute error is $|y_{\text{exact}}(t) - y_N(t)|$. The relative error is $\frac{E_{abs}(t)}{|y_{exact}(t)|}$, $y_{exact}(t) \neq 0$. The maximum absolute error on the tabulated grid is $max_i E_{\text{abs}}(t_i)$. For method-level summaries, the mean absolute error is $\frac{1}{n}\sum_{i=1}^{n}|y_{exact}(t_i) - y_N(t_i)|$, and the root-mean-square error is $\sqrt{\frac{1}{n}\sum_{i=1}^{n}[y_{exact}(t_i) - y_N(t_i)]^2}$. Three error measures are used. The absolute error is $E_{abs}(t) = \left|y_{exact} - y_{N(t)}\right|$. The relative error is $E_{rel(t)} = \frac{E_{abs(t)}}{|y_{exact(t)}|}$, when $y_{exact(t)} \neq 0$. The maximum absolute error on the tabulated grid is $E_{max} = max_i\, E_{abs(t_i)}$. For method-level summaries, the mean absolute error and root-mean-square error are also reported. These measures separate local pointwise agreement from overall performance on the selected grid.

DTM Solutions of Linear and Nonlinear Differential Equations

In this section, DTM is applied to representative linear and nonlinear differential equations. The approximate solutions are compared with exact solutions, and the numerical accuracy is evaluated using the error measures defined above.

Problem 1

Linear differential equation of order 2, (Ngwane & Jator, 2017).

$$y'' = -10y + 99Sint\} \tag{4}$$

Where the initial conditions are $y(0) = 1, y'(0) = 11$. The exact solution is given as:

$$y(t) = sin\, t + cos\, 10t + sin\, 10t \tag{5}$$

The transform of equation (4) is given below:

$$\frac{(k+2)!}{k!}Y(k+2) = -100Y(k) + 99\left\{\frac{1}{k!}sin(\frac{k\pi}{2})\right\}.$$

$$Y(1) = 11, Y(0) = 1,$$

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left\{-100Y(k) + 99\frac{1}{k!}\left(\sin\left(\frac{k\pi}{2}\right)\right)\right\}. \quad (6)$$

The DTM approximated solution for Equation (4) is given below:

$$y(t) = 1 + 11t - 50t^2 - \frac{1001t^3}{6} + \frac{1250t^4}{3} + \frac{100001t^5}{120} - \frac{12500t^6}{9} - \frac{10000001t^7}{5040} + \frac{156250t^8}{63} + \frac{142857143t^9}{51840} - \frac{1562500t^{10}}{567}$$

Table 1 displays the exact, DTM solutions and the absolute error. Figure 1 shows the comparison of the DTM approximation with the exact solution for Problem 1. The table depicts the corresponding point to point absolute errors. The close agreement over the stated interval is dependable with the small truncation remainder for the preferred expansion order.

Table 1: Exact, DTM solutions Abs Error

t	Exact solution	DTM solution	Abs Error
0	1	1	0
0.01	1.1048	1.1048	0
0.02	1.1987	1.1987	4.4408×10^{-16}
0.03	1.2808	1.2808	4.3298×10^{-14}
0.04	1.3504	1.3504	1.0145×10^{-12}
0.05	1.4069	1.4069	1.1703×10^{-11}
0.06	1.4499	1.4499	8.6143×10^{-11}
0.07	1.479	1.479	4.6499×10^{-10}
0.08	1.4939	1.4939	2.0002×10^{-9}
0.09	1.4948	1.4948	1.0032×10^{-9}
0.1	1.4816	1.4816	2.2816×10^{-8}

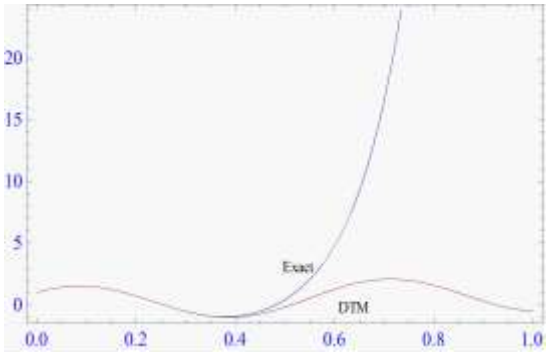


Figure 1: Exact and DTM solutions

For Problem 1, a tenth-ordered DTM polynomial is used on $0 \leq t \leq 0.1$. The largest tabulated absolute error is approximately 1×10^−8; the corresponding maximum relative error is approximately 6.7×10^−9 based on the tabulated exact values.

Problem 2

Problem 2 has previously been treated using OHAM, HAM, and HPM (Naeem et al., 2016). The same initial-value problem is considered here with DTM so that the published approximations can be compared with the exact solution. Applying DTM to Problem 2 gives the coefficient recurrence shown below and the corresponding truncated series solution.

$$y''(t) + 2y'(t) + y(t) = 0, \quad (7)$$

Whereas, $y(0) = 1$, and $y'(0) = 0$. The exact solution of the problem is:

$$y(t) = te^{-t} + e^{-t}. \quad (8)$$

Equation (7) is transformed by applying DTM:

$$\frac{(k+2)!}{k!}Y(k+2) + 2(k+1)Y(k+1) + Y(k) = 0, \quad (9)$$

$$Y(0) = 1, and Y'(0) = 0.$$

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\big(-(2(k+1)Y(k+1) + Y(k))\big).$$

$$\left\{1,0,-\frac{1}{2},\frac{1}{3},-\frac{1}{8},\frac{1}{30},-\frac{1}{144},\frac{1}{840},-\frac{1}{5760},\frac{1}{45360},-\frac{1}{403200}\right\}.$$

DTM solution is of the problem 2 is obtained, which is given below:

$$y(t) = 1 - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{8} + \frac{t^5}{30} - \frac{t^6}{144} + \frac{t^7}{840} - \frac{t^8}{5760} \quad (10)$$

The OHAM result:

$$y_{oham}(t) = 1 - 0.424995t^2 - 0.161002t^3.$$

HPM result:

$$y_{hpm}(t) = 1 - 0.5t^2 - 0.33333t^3 + 0.0417t^4,$$

and HAM result:

$$y_{ham}(t) = 1 - 0.0.464612t^2 + 0.209414t^3 - 0.00742224\,t^4 - 0.00671979\mathrm{t}^5.$$

Tabulated results of OHAM exact solutions and their absolute (abs) error is presented in Table 2, while the numerical comparison of the exact with HPM solution is shown in Table 3.

Table 2: Exact, OHAM solutions and absolute (Abs) error

t	Exact solution	OHAM solution	Abs Error
0	1	1	0.
0.1	0.99532	0.99590	5.9×10^{-4}
0.2	0.98247	0.98428	1.8×10^{-3}
0.3	0.96306	0.96629	3.0×10^{-3}
0.4	0.93844	0.94230	3.8×10^{-3}
0.5	0.90979	0.92387	4.0×10^{-3}
0.6	0.87809	0.88177	3.6×10^{-3}
0.7	0.84419	0.84697	2.7×10^{-3}
0.8	0.80879	0.81043	1.6×10^{-3}
0.9	0.77248	0.77312	6.4×10^{-4}
1	0.73575	0.73600	2.4×10^{-4}

Table 3: Exact and HPM solutions

t	Exact solution	HPM solution	Abs Error
0	1	1	0.
0.1	0.9953	0.9953	1.6×10^{-5}
0.2	0.9824	0.9827	2.5×10^{-5}
0.3	0.9630	0.9643	1.2×10^{-3}
0.4	0.9384	0.9424	3.9×10^{-3}
0.5	0.9097	0.9192	9.4×10^{-3}
0.6	0.8780	0.8974	1.9×10^{-2}
0.7	0.8441	0.8793	3.5×10^{-2}
0.8	0.8087	0.8677	5.8×10^{-2}
0.9	0.7724	0.8653	9.2×10^{-2}
1	0.7357	0.875	1.3×10^{-1}

Comparative analysis of HAM with exact solution is shown in the Table 4. The absolute error is the difference between the exact and HAM solution. In first three iterations the result is relatively good as compared with the rest of iterations.

Table 4: Exact and HAM solutions

`	Exact solution	HAM solution	Abs Error
0	1	1	0.
0.1	0.99532	0.99556	5.9×10^{-4}
0.2	0.98247	0.98307	1.8×10^{-3}
0.3	0.96306	0.96376	3.0×10^{-3}
0.4	0.93844	0.93880	3.8×10^{-3}
0.5	0.90979	0.90935	4.0×10^{-3}
0.6	0.87809	0.87648	3.6×10^{-3}
0.7	0.84419	0.84125	2.7×10^{-3}
0.8	0.80879	0.80462	1.6×10^{-3}
0.9	0.77248	0.76749	6.4×10^{-4}
1	0.73575	0.73066	2.4×10^{-4}

Table 5: Exact, DTM Solutions and Abs Error

t	Exact solution	DTM solution	Abs Error
0	1	1	0.
0.1	0.99532	0.99532	1.1102×10^{-16}
0.2	0.98247	0.98247	4.8849×10^{-15}
0.3	0.96306	0.96306	4.3187×10^{-13}
0.4	0.93844	0.93844	1.0135×10^{-11}
0.5	0.90979	0.90979	1.1694×10^{-10}
0.6	0.87809	0.87809	8.6129×10^{-10}
0.7	0.84419	0.84419	4.6534×10^{-9}
0.8	0.80879	0.80879	2.0041×10^{-8}
0.9	0.77248	0.77248	7.2588×10^{-8}
1	0.73575	0.73575	2.2934×10^{-7}

Table 5 (above) compares the tenth-order DTM approximation with the exact solution. The maximum absolute error on $0 \leq t \leq 1$ is 2.2934×10^{-7}, corresponding to a maximum relative error of approximately 3.12×10^{-7}. This high-order result should not be interpreted as a direct proof that DTM is universally more accurate than HAM, HPM, or OHAM

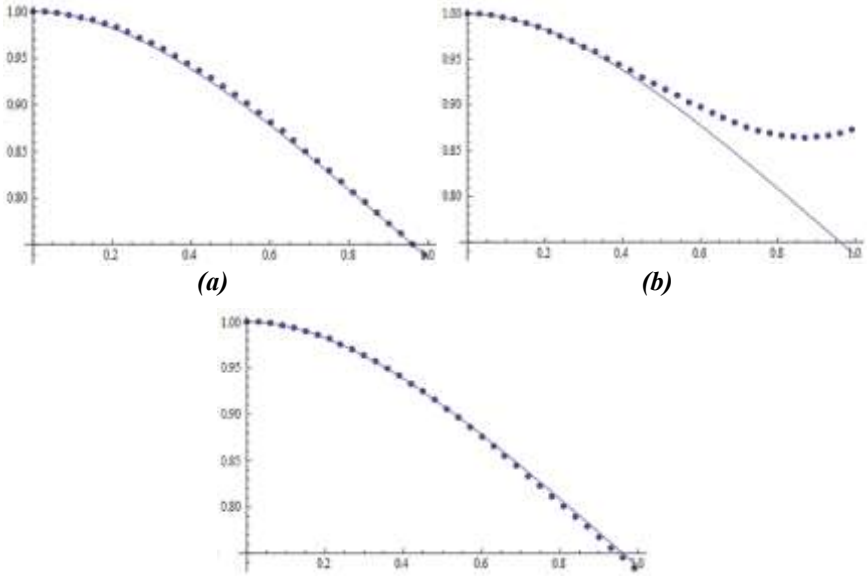
because the published comparator formulas use different approximation structures and orders.

Validation of the Method

Table 6 summarizes the pointwise errors described for the published work (Naeem et al., 2016) and for the tenth-ordered DTM solution. The results show a strong agreement with high-order DTM series and the exact solution; however, the methods are not for the same polynomial order in that study. A separate order-normalized comparison is provided below.

Table 6: Error analysis for OHAM, HPM, Naeem et al. (2016) HAM and DTM

T	OHAM error	HPM error	HAM error	DTM error
0	0.	0.	0.	0.
0.1	5.9×10^{-4}	1.6×10^{-5}	5.9×10^{-4}	1.110×10^{-16}
0.2	1.8×10^{-3}	2.5×10^{-5}	1.8×10^{-3}	4.884×10^{-15}
0.3	3.0×10^{-3}	1.2×10^{-3}	3.0×10^{-3}	4.318×10^{-13}
0.4	3.8×10^{-3}	3.9×10^{-3}	3.8×10^{-3}	1.013×10^{-11}
0.5	4.0×10^{-3}	9.4×10^{-3}	4.0×10^{-3}	1.169×10^{-10}
0.6	3.6×10^{-3}	1.9×10^{-2}	3.6×10^{-3}	8.612×10^{-10}
0.7	2.7×10^{-3}	3.5×10^{-2}	2.7×10^{-3}	4.6534×10^{-9}
0.8	1.6×10^{-3}	5.8×10^{-2}	1.6×10^{-3}	2.0041×10^{-8}
0.9	6.4×10^{-4}	9.2×10^{-2}	6.4×10^{-4}	7.2588×10^{-8}
1	2.4×10^{-4}	1.3×10^{-1}	2.4×10^{-4}	2.2934×10^{-7}



(a) ***(b)*** ***(c)***

Figure 2 (Naeem et al., 2016): (a) Graph of the exact and OHAM solution; (b) Graph of the exact and HPM solution; (c) Graph of the exact and HAM solution

Order-Normalized Comparison for Problem 2

To address differences in approximation order, the OHAM, HAM, HPM, and DTM are truncated to the common polynomial degree $N = 3$, and evaluated on the same grid $t = 0, 0.1, \ldots, 1.0$. The resulting errors are shown in Table 7. This comparison reveals that no method is uniformly superior solely because of its label: at $N = 3$, the HAM polynomial has the smallest error for this problem, whereas the DTM error decreases rapidly, when its systematic recurrence is continued to higher order. Accordingly, the manuscript confines comparative claims to the stated order and accuracy conditions.

Table 7: Order-normalized error comparison for Problem 2 on t=0,0. 1, 1.0.

Method	Common degree N	MAE	RMSE	Maximum absolute error
OHAM	3	8.6566×10^{-2}	1.3569×10^{-1}	3.2176×10^{-1}
HPM	3	1.6016×10^{-1}	2.4429×10^{-1}	5.6909×10^{-1}
HAM	3	1.5083×10^{-3}	3.0031×10^{-3}	9.0431×10^{-3}
DTM	3	2.3173×10^{-2}	3.8669×10^{-2}	9.7574×10^{-2}

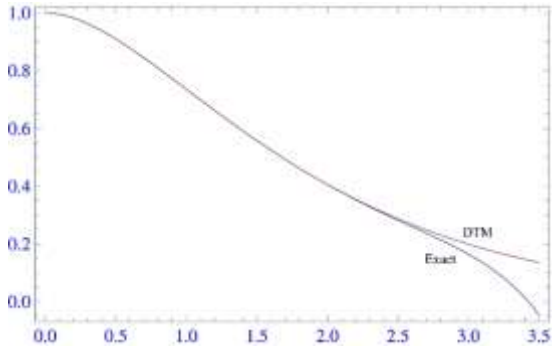


Figure 3: Exact and DTM Solutions

Figure 3 shows the exact and DTM curves for Problem 2, while Figures 2(a-c) display the published OHAM, HPM, and HAM comparisons. Because the methods in those figures use different approximation constructions, the figures are used as qualitative illustrations rather than as evidence of a universal convergence-rate ranking.

Problem 3

This problem examines a fourth-order nonlinear ordinary differential equation with a known exact solution (Polyanin & Shingareva, 2019). It is included to test the coefficient recurrence for a nonlinear, higher-order model.

$$\left.\begin{array}{l} y_t^{''''}(t) = y^2(t)y_t^{''}(t) + 12y^5(t), \\ y(0) = 1, y_t^{'}(0) = 1, \\ y_{tt}^{"}(0) = 2, y_t^{'''}(0) = 6. \end{array}\right\} \quad (11)$$

Where the exact solution is $\frac{1}{1-t}$.

The transformed form of the problem by utilizing DTM

$$\frac{(k+4)!}{k!}Y(k+4) =$$

$$\left\{\sum_{s=0}^{k}\sum_{m=0}^{k-s}\begin{array}{l}(1+k-m-s)(2+k-m-s) \\ Y(2+k-m-s)Y(m)Y(s) + 12\delta(k-12)\end{array}\right\}$$

$$Y(1) = Y(2) = Y(3) = Y(0) = 1.$$

$$Y(k+4) = \frac{1}{(k+1)(k+2)(k+3)(k+4)} \times$$

$$\left\{\sum_{s=0}^{k}\sum_{m=0}^{k-s}\begin{array}{l}(1+k-m-s)(2+k-m-s) \\ Y(2+k-m-s)Y(m)Y(s) + 12\delta(k-12)\end{array}\right\}$$

$$Y(1) = 2.Y(0) = 1.Y(2) = 3.Y(3) = 1.$$

DTM coefficients of the problem $\{1,1,1,1,\frac{1}{4},\frac{3}{10},\frac{1}{3},\frac{19}{70},\frac{31}{140},\frac{307}{1680},\frac{1837}{12600}\}$

The series solution of the problem 3 is given by:

$$y(t) = 1 + t + t^2 + t^3 + \frac{t^4}{4} + \frac{3t^5}{10} + \frac{t^6}{3} + \frac{19t^7}{70} + \frac{31t^8}{140} + \frac{307t^9}{1680} + \frac{1837t^{10}}{12600}$$

Table 8 compares the exact and DTM solutions for Problem 3 and reports the pointwise absolute error. The table displays close agreement on the reported interval. The highes absolute error is 8.2748×10^{-5} at t=0.1, corresponding to a relative error 7.45×10^{-5} based on the exact value.

Table 8: Exact and DTM Solutions

t	Exact solution	DTM solution	Abs error
0	1	1	0.
0.01	1.0101	1.0101	7.5706×10^{-9}
0.02	1.0204	1.0204	1.2228×10^{-7}
0.03	1.0309	1.0309	6.2501×10^{-7}
0.04	1.0416	1.0416	1.9945×10^{-6}
0.05	1.0526	1.0526	4.9172×10^{-6}
0.06	1.0638	1.0638	1.0297×10^{-5}
0.07	1.0752	1.0752	1.9268×10^{-5}
0.08	1.0869	1.0869	3.3205×10^{-5}
0.09	1.0988	1.0989	5.3733×10^{-5}
0.1	1.1110	1.1111	8.2748×10^{-5}

Figure 4 displays the graphical comparison between the DTM and the exact solution for Problem 3. The plot is consistent with the tabularized errors and shows the local accuracy of the truncated series.

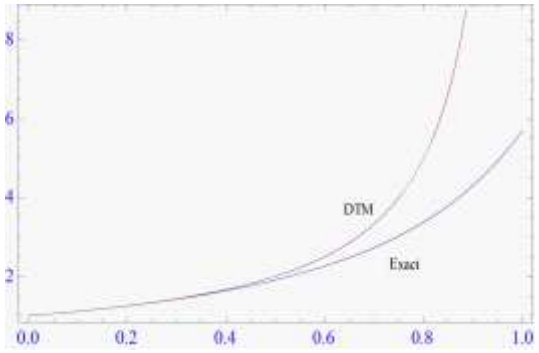


Figure 4: Exact and DTM Solutions.

Problem 4 (Nonlinear Riccati Equation with a Finite Radius of Convergence)

To further examine the nonlinear behavior of the method and highlight the importance of analyticity, consider the following nonlinear differential equation:

$y'(t) = 1 + y^2(t)$, subject to the initial condition $y(0) = 0$.

The exact solution of this problem is $t = \pm\frac{\pi}{2}$.

For this solution, the singularities nearest to the point $(t = 0)$ are located, therefore, the Taylor series, and the DTM series, converges within $R = \frac{\pi}{2}$.

Using the Differential Transform Method, the leads to the recurrence relation $\delta(k,0) + \sum_{r=0}^{k} Y(r)Y(k-r)$, together with $Y(0) = 0$.

By successively evaluating the recurrence relation, the nonzero transformed coefficients up to the thirteenth order are obtained as:

$$Y(1) = 1, Y(3) = \frac{1}{3}, Y(5) = \frac{2}{15}, Y(7) = \frac{17}{315}, Y(9) = \frac{62}{2835},$$
$$Y(11) = \frac{1382}{155925}, Y(13) = \frac{21844}{6081075}.$$

Since all even-order coefficients vanish, the thirteenth-order DTM approximation can be written as

$$y(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} + \frac{21844}{6081075}t^{13}.$$

This thirteenth-order approximation is used in the following analysis to study the convergence behavior of the DTM solution and to compare with the exact solution.

The error remains very near the expansion point, but increases as t approaches the singularity at π/2. On $0 \le t \le 1$, the highest absolute error is 2.45×10^{-3} and the maximum relative error is approximately 1.57×10^{-3}. This behavior illustrates the dependence of DTM on truncation order and radius of convergence.

Results and Discussion

From problems 1-3, the DTM results agree very close with the exact solutions on the particular intervals, as depicted in tables 1, 5, and 8 and pictured in Figures 1, 3, and 4. The revised error assessment adds maximum, relative and absolute errors to the original pointwise absolute-error tables. For Problem 2, the tenth-order DTM approximation has a highest absolute error of 2.2934×10^{-7} on $0 \leq t \leq 1$. While, Table 7 shows, that when all methods are restricted to the same cubic degree, HAM has the smallest error for that specific comparison. Thus, the data support high accuracy of the higher-order DTM recurrence but do not support an unconditional claim that DTM is always more accurate than HAM, HPM, or OHAM.

The added Riccati test in Table 9 delivers a more challenging nonlinear example and also exposures a restriction that is not visible in the first three tests. Because tan(t) has singularities at ±π/2, its DTM series about $t = 0$, is essentially local. The increasing error at $t = 1$ is consistent with the convergence bound, stated earlier. This result shows that DTM is most consistent, when the solution is analytic over a suitably large neighborhood of the expansion point, while, when the truncation order is selected relative to the intended interval.

Table 9: Exact and thirteenth-order DTM solutions for the nonlinear Riccati problem.

T	Exact tan(t)	DTM (N=13)	Absolute error	Relative error
0.0	0.000000	0.000000	0.000	0.000
0.1	0.100335	0.100335	2.776×10^{-17}	2.766×10^{-16}
0.2	0.202710	0.202710	4.852×10^{-14}	2.393×10^{-13}
0.3	0.309336	0.309336	2.168×10^{-11}	7.009×10^{-11}
0.4	0.422793	0.422793	1.672×10^{-9}	3.954×10^{-9}
0.5	0.546302	0.546302	4.944×10^{-8}	9.049×10^{-8}
0.6	0.684137	0.684136	8.014×10^{-7}	1.171×10^{-6}
0.7	0.842288	0.842280	8.624×10^{-6}	1.024×10^{-5}
0.8	1.029639	1.029569	6.916×10^{-5}	6.717×10^{-5}
0.9	1.260158	1.259712	4.462×10^{-4}	3.541×10^{-4}
1.0	1.557408	1.554960	2.448×10^{-3}	1.572×10^{-3}

The figures and tables are therefore interpreted together: the plots show qualitative agreement, while the error tables quantify the differences. DTM remains attractive because of its recurrence relations are reliable to produce and extend systematically to high order. Nevertheless, computational efficiency depends on the equation, symbolic algebra required by nonlinear products, desired tolerance, and the need for possible domain decomposition or multi-step continuation.

Conclusion

This study evaluated the Differential Transform Method through linear, nonlinear, and higher-order differential-equation problems by means of exact-solution comparisons and a more explicit validation framework. The revised analysis confirms the study's contribution by adding a convergence-based truncation estimate, a stopping criterion, relative and maximum absolute errors, and an order-normalized comparison with HAM, HPM, and OHAM. The results clarify that DTM can achieve high accuracy, when the solution is analytic in the region of interest, and when a sufficient truncation order is taken into account. The nonlinear Riccati example further shows, that how the error grows as the evaluation point approaches the nearest singularity. DTM is mainly limited by the fact that its power-series solution is valid only within a certain region. The accuracy of DTM may decrease near singularities, discontinuities, sharp gradients, or other non-smooth regions. In addition, using higher-order approximations for strongly nonlinear systems can increase the computational effort and memory required. For problems whose solution has a limited radius of convergence, multi-step DTM, analytic continuation, or hybrid numerical strategies may be required. Overall, these results show that DTM is a reliable and reproducible semi-analytical method, although it should not be considered the best choice for every type of differential equation

References

Ali, J. (2012). for Some Higher Order Boundary Value Problems Differential Transform Method (DTM). *Int. J. Contemp. Math. Sciences*, *7*(6), 263–272.

AM, K. (2018). Numerical Technique for a System of Nonlinear Partial Differential Equations of the Time-Fractional Order. *Journal of Applied & Computational Mathematics*, *07*(02).

Applications, C. (2010). Analytical Solution to determine displacement of Nonlinear oscillations with Parametric Excitation by Differential Transformation Method. *Mathematical and Computational Applications*, *15*(5), 810–815.

Benhammouda, B., Vazquez-leal, H., Hernandez-martinez, L., Instrumentation, E., Veracruzana, U., Gonzalo, C., & Beltr, A. (2014). Modified Differential Transform Method for Solving the Model of Pollution for a System of Lakes. *Hindawi Publishing Corporation Discrete Dynamics in Nature and Society*, *2014*.

Carr, P. (2006). Add to my library Features : *Implementing Culture Change*, *0404*, 114.

El-Gamel, M., & Abd El-Hady, M. (2013). Computation of the

eigenvalues and eigenfunction of generalized Sturm-Liouville problems via the differential transformation method. *International Journal of Research and Reviews in Applied Sciences* (*IJRRAS*), *15*(2), 259–268.

Ertürk, V. S., & Momani, S. (2008). Differential transform technique for solving fifth-order boundary value problems. *Mathematical and Computational Applications*, *13*(2), 113–121.

Hashim, I., Noorani, M. S. M., & Said Al-Hadidi, M. R. (2006). Solving the generalized Burgers-Huxley equation using the Adomian decomposition method. *Mathematical and Computer Modelling*, *43*(11–12), 1404–1411.

Lu, M. O. Ğ., Kurulay, M., & Bayram, M. (2008). Solution of the Differential-Algebraic Equations With Index 3. *Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi,* 25–30.

Naeem, M. M., Jan, P. R. U., & Haq, A. (2016). Solving different order boundary value problems using optimal homotopy asymptotic method. *Science International, 28*(2), 803–809.

Nawaz, Y. (2011). Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. *Computers and Mathematics with Applications*, *61*(8), 2330–2341.

Ngwane, F. F., & Jator, S. N. (2017). A Trigonometrically Fitted Block Method for Solving Oscillatory Second-Order Initial Value Problems and Hamiltonian Systems. *International Journal of Differential Equations*, *2017*.

Odibat, Z., & Momani, S. (2008). A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters*, *21*(2), 194–199.

On the two-dimensional differential transform method. (2003). *Applied Mathematics and Computation*, *143*(2–3), 361–374.

Polyanin, A. D., & Shingareva, I. K. (2019). Non-linear blow-up problems for systems of ODEs and PDEs: Non-local transformations, numerical and exact solutions. *International Journal of Non-Linear Mechanics*, *111*, 28–41.

Polyanin, A. D., & Zaitsev, V. F. (2017). Handbook of ordinary differential equations: Exact solutions, methods, and problems. In *Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems* (Vol. 2017, Issue January).

R. Bosede, O. (2012). On Some Numerical Methods for Solving Initial Value Problems in Ordinary Differential Equations. *IOSR Journal of Mathematics*, *1*(3), 25–31.

Rawashdeh, M. (2013). Using the Reduced Differential Transform

Method to solve nonlinear PDEs Arises in biology and physics. *World Applied Sciences Journal*, *23*(8), 1037–1043.

Siddiqi, S. S., Akram, G., & Iftikhar, M. (2012). Solution of seventh order boundary value problems by variational iteration technique. *Applied Mathematical Sciences*, *6*(93–96), 4663–4672.

U, S. neelam. (2014). *Review of literature 2 . Review of literature*. *Amjad 2011*, 49–58.

Ullah, I., Rahim, M. T., Khan, H., & Qayyum, M. (2016). Analytical analysis of squeezing flow in porous medium with mhd effect. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, *78*(2), 281–292.

Warade, N. O., & Rastogi, P. (2018). Application to Differential Transformation Method for Solving Eight Order Ordinary Differential Equations. *International Journal of Mathematics Trends and Technology*, *61*(3), 177–185.

Xu, L. (2007). Variational iteration method for solving integral equations. *Computers and Mathematics with Applications*, *54*(7–8), 1071–1078.

Yazdani, A., Vahidi, J., & Ghasempour, S. (2016). Comparison Between Differential Transform Method and Taylor Series Method for solving Linear and Nonlinear Ordinary Differential Equations. *International Journal of Mechatronics,Electrical and Computer Technology*, *6*(20), 2872–2877. http://www.aeuso.org